A PAIR CORRELATION HYPOTHESIS AND THE EXCEPTIONAL SET IN GOLDBACH'S PROBLEM

A. LANGUASCO AND A. PERELLI

§1. *Introduction.* In 1973, Montgomery [**12**] introduced, in order to study the vertical distribution of the zeros of the Riemann zeta function, the pair correlation function

$$F(X,T)=\sum_{0<\gamma_1,\gamma_2\leqslant T} X^{i(\gamma_1-\gamma_2)}w(\gamma_1-\gamma_2),$$

where $w(u)=4/(4+u^2)$ and γ_j, $j=1,2$, run over the imaginary part of the non-trivial zeros of $\zeta(s)$. It is easy to see that, for $T\to\infty$,

$$F(X,T)\ll T\log^2 T$$

uniformly in X, and Montgomery [**12**], see also Goldston–Montgomery [**7**], proved that under the Riemann Hypothesis (RH)

$$F(X,T)\sim\frac{1}{2\pi}T\log T \tag{1}$$

uniformly for $X\leqslant T\leqslant X^A$, for any fixed $A>1$. He also conjectured, under RH, that (1) holds uniformly for $X^{\varepsilon}\leqslant T\leqslant X$, for every fixed $\varepsilon>0$. We denote by MC the above conjecture.

It is well known that MC is strongly connected with the distribution of primes and related problems, see, *e.g.*, Gallagher–Mueller [**3**], Heath-Brown [**9**], Goldston Montgomery [**7**] and Goldston [**4**]. In particular, Goldston–Montgomery [**7**] showed that MC is equivalent to a certain asymptotic formula for the mean-square of primes in short intervals, and Goldston [**4**] deduced from MC the existence of Goldbach numbers in short intervals.

Our aim here is to study the size of the exceptional set in Goldbach's problem under the assumption of the Generalized Riemann Hypothesis (GRH) and a pair correlation type hypothesis for Dirichlet L-functions. For $(a,q)=1$ write

$$F(X,T;q,a)=\sum_{\chi_1,\chi_2\ (\mathrm{mod}\ q)}\chi_1(a)\bar{\chi}_2(a)\tau(\bar{\chi}_1)\tau(\chi_2)\times\sum_{|\gamma_1|,|\gamma_2|\leqslant T}X^{i(\gamma_1-\gamma_2)}w(\gamma_1-\gamma_2),$$

where $\tau(\chi)$ denotes the Gauss sum and γ_j, $j=1,2$, run over the imaginary part of the non-trivial zeros of $L(s,\chi_j)$. The trivial bound for $F(X,T;q,a)$, as $T\to\infty$, is clearly

$$F(X,T;q,a)\ll q\varphi(q)^2T\log^2 qT$$

uniformly in X, q and a. By adapting Montgomery's method in [**12**] we obtain

THEOREM 1. *Assume* GRH *and let* $A>1$ *by any fixed constant. Then*

$$F(X, T; q, a) \sim \frac{1}{\pi} \varphi(q)^2 T \log X \tag{2}$$

uniformly for $X \log X \leqslant T \leqslant X^A$ *and* $q \leqslant X \log^{-4} X$.

We omit the proof of Theorem 1 since it follows *mutatis mutandis* from the argument in [**12**]. We remark that in the same way we can also obtain the estimate

$$F(X, T; q, a) \ll_A q^2 T \log X \tag{3}$$

uniformly for $X \leqslant T \leqslant X^A$ and $q \leqslant X \log^{-3} X$.

In view of Theorem 1 and (3) we consider the following hypothesis. Assume GRH and let $\theta \in (0, \frac{1}{2}]$ be fixed and $V = X^{1-\theta}/q$; then for every $\varepsilon > 0$

$$F(X, T; q, a) \ll_\varepsilon q^2 T X^\varepsilon \tag{4}$$

uniformly for $V \leqslant T \leqslant X$, $q \leqslant X^\theta$ and $(a, q) = 1$. We denote by GMC(θ) the above hypothesis. Comparing MC, (2), (3) and (4) we see that GMC(θ) may be regarded as a weak form of a pair correlation hypothesis for a suitable average of zeros of L-functions. However, we remark that GMC(θ) is a rather strong hypothesis on the cancellation with respect to q in $F(X, T; q, a)$.

The connection between GMC(θ) and the exceptional set in Goldbach's problem is provided by the following two results. Let $(a, q) = 1$, $Q > 1$, $H \leqslant X$,

$$E(X, H) = |\{2n \in [X, X+H] : 2n \text{ is not a sum of two primes}\}|,$$

$$E(X) = E(X, X), \qquad S(\alpha) = \sum_{\frac{1}{4}X \leqslant n \leqslant 2X} \Lambda(n) e(n\alpha), \qquad T(\alpha) = \sum_{\frac{1}{4}X \leqslant n \leqslant 2X} e(n\alpha),$$

$$R(\eta; q, a) = S\left(\frac{a}{q} + \eta\right) - \frac{\mu(q)}{\varphi(q)} T(\eta)$$

and

$$I(X, Q; q, a) = \int_{-1/qQ}^{1/qQ} |R(\eta; q, a)|^2 d\eta.$$

By an adaptation of the method of Kaczorowski–Perelli–Pintz [**10**] we get

THEOREM 2. *Assume* GRH. *Let* $\varepsilon > 0$ *and* $X^\varepsilon \leqslant Q \leqslant \frac{1}{2} X^{1/2}$. *Assume also that*

$$I(X, Q; q, a) \ll_\varepsilon \frac{X^{1+\varepsilon}}{qQ} \tag{5}$$

uniformly for $q \leqslant Q$ *and* $(a, q) = 1$. *Then* $E(X, 4Q^2) \ll_\varepsilon X^\varepsilon$.

We take this opportunity to correct a mistake occurring in [**10**], which has been kindly pointed out by Professor R. C. Vaughan. This is done in Section 5. Here we point out that, after correction of that mistake, the method in [**10**] yields the bound $I(X, Q; q, a) \ll Q^{-1}X\log^4 X$ uniformly for $q \leqslant Q \leqslant X$ and $(a, q) = 1$ under the assumption of GRH, which is weaker than (5) for large q.

The next step is to link $I(X, Q; q, a)$ with GMC(θ). Such a link is provided by the following

THEOREM 3. *Let $\theta \in (0, \frac{1}{2}]$ be fixed, $Q = \frac{1}{2}X^\theta$ and assume* GMC(θ). *Then* (5) *holds uniformly for $q \leqslant Q$ and $(a, q) = 1$.*

From Theorems 2 and 3 we obtain

COROLLARY 1. *Let $\theta \in (0, \frac{1}{2}]$ be fixed. Then* GMC(θ) *implies that $E(X, X^{2\theta}) \ll_\varepsilon X^\varepsilon$ and $E(X) \ll_\varepsilon X^{1-2\theta+\varepsilon}$.*

The second assertion of Corollary 1 follows immediately from the first, by subdividing the interval $[X, 2X]$ into $\ll X^{1-2\theta}$ intervals of the form $[Y_j, Y_j + Y_j^{2\theta}]$ with suitable $X \leqslant Y_j \leqslant 2X$.

It is well known that GRH implies that $E(X) \ll_\varepsilon X^{1/2+\varepsilon}$, see Hardy–Littlewood [**8**], and hence GMC(θ) appears to have a significant relevance to $E(X)$ only if $\theta \in (\frac{1}{4}, \frac{1}{2}]$. In fact, the method in [**8**] can be used to obtain a more direct link between GMC(θ) and $E(X)$, instead of arguing *via* $E(X, H)$ as in Corollary 1. In this way we can get the same bound for $E(X)$ as in Corollary 1, but we lose information on $E(X, H)$.

As it will be clear from (6) below, in order to use the method of Hardy–Littlewood [**8**] we need, essentially, an individual estimate for $S(\alpha)$ over suitable minor arcs. To this end, we introduce the following slight extension of GMC(θ). Assume GRH, let $\theta \in (0, \frac{1}{2}]$ be fixed, $\xi = \xi(X, q) \in (0, \frac{1}{2}]$ and $W = \frac{1}{2}X\xi$; then for every $\varepsilon > 0$

$$F(X, T; q, a) \ll_\varepsilon q^2 T X^\varepsilon$$

uniformly for $W \leqslant T \leqslant X$, $q \leqslant X^\theta$ and $(a, q) = 1$. We denote by GMC($\theta; \xi$) this hypothesis. We observe that GMC($\theta; \xi$) coincides with GMC(θ) if $\xi = 1/qQ$ and $Q = \frac{1}{2}X^\theta$. We have

THEOREM 4. *Let $\theta \in (0, \frac{1}{2}]$ be fixed, ξ be as above and assume* GMC(θ, ξ). *Then for every $\varepsilon > 0$*

$$S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + O_\varepsilon(X^{1/2+\varepsilon}(1 + (X\xi)^{1/2}))$$

uniformly for $q \leqslant X^\theta$, $(a, q) = 1$ and $|\eta| \leqslant \xi$.

Theorem 4 may be compared with the result obtained by Baker–Harman [**1**], Lemma 12, assuming only GRH, *i.e.*,

$$S\left(\frac{a}{q} + \eta\right) \ll \left(\frac{|\mu(q)|}{q \log X} \min(X, |\eta|^{-1}) + (qX)^{1/2}\left(1 + \left(\frac{|\eta| X}{\log X}\right)^{1/2}\right)\right) \log^2 X$$

for $|\eta| \leqslant X^{-1/2}$. Essentially, our result saves a factor $q^{1/2}$ with respect to the Baker–Harman result, and this reflects the analogous saving assumed in

GMC(θ; ξ). However, we need to use a more sophisticated technique than that in [**1**] in order to exploit GMC(θ; ξ). We also remark that, assuming only GRH, the method of Baker–Harman [**1**], as optimized in Lemma 5 of Goldston [**5**], apparently yields a slightly better result than that obtainable, under the same assumption, by the method used in the proof of Theorem 4.

Since Theorem 4 can be proved along the lines of Theorem 3, we will give only a brief sketch of its proof. From Theorem 4 and the above observation we immediately get

COROLLARY 2. *Let* $\theta \in (0, \frac{1}{2}]$ *be fixed,* $Q = \frac{1}{2}X^{\theta}$ *and assume* GMC(θ). *Then for every* $\varepsilon > 0$

$$S\left(\frac{a}{q}+\eta\right)=\frac{\mu(q)}{\varphi(q)}T(\eta)+O_{\varepsilon}\left(\frac{X^{1+\varepsilon}}{(qQ)^{1/2}}\right)$$

uniformly for $q \leqslant Q$, $(a, q) = 1$ *and* $|\eta| \leqslant 1/qQ$.

Consider now the Farey dissection of the unit interval of order $Q = \frac{1}{2}X^{\theta}$ and denote by $\mathfrak{M}(q, a)$ the Farey arc at a/q. Choose $P = Q\log^{-10} X$ and consider the major and minor arcs relative to P and Q, *i.e.*,

$$\mathfrak{M}(\theta)=\bigcup_{q\leqslant P}\ \bigcup_{(a,q)=1}\mathfrak{M}(q,a) \qquad \text{and} \qquad \mathfrak{m}(\theta)=\bigcup_{P<q\leqslant Q}\ \bigcup_{(a,q)=1}\mathfrak{M}(q,a).$$

The key point in [**8**] is to obtain a suitable bound for $\int_{\mathfrak{m}(\theta)}|S(\alpha)|^4 d\alpha$. From Corollary 2 we obtain

$$\int_{\mathfrak{m}(\theta)}|S(\alpha)|^4 d\alpha \ll \sup_{\alpha\in\mathfrak{m}(\theta)}|S(\alpha)|^2\int_0^1|S(\alpha)|^2 d\alpha \ll_{\varepsilon} X^{3-2\theta+\varepsilon} \tag{6}$$

and hence the method of Hardy–Littlewood [**8**] can be shown to yield again $E(X) \ll_{\varepsilon} X^{1-2\theta+\varepsilon}$, under GMC($\theta$).

We observe that, in the extreme case $\theta = \frac{1}{2}$, from Corollaries 1 and 2 we get the "best possible" results

$$E(X) \ll_{\varepsilon} X^{\varepsilon}$$

and

$$\sup_{\alpha\in\mathfrak{m}(\frac{1}{2})}|S(\alpha)| \ll_{\varepsilon} X^{1/2+\varepsilon},$$

of course under GMC($\frac{1}{2}$), to be compared with Goldston [**5**] and [**6**].

We finally point out that the X^{ε} in all previous results can be replaced by a suitable power of $\log X$, by assuming a form of GMC(θ) and GMC(θ; ξ) with a power of $\log X$ instead of X^{ε}. Moreover, in the proofs below we will denote by ε an arbitrarily small positive constant, whose value will not necessarily be the same at each occurrence.

§2. *Proof of Theorem* 2. Given $Q \in [X^{\varepsilon}, \frac{1}{2}X^{1/2}]$, let $H = 4Q^2$, $P = Q\log^{-10} X$, $\mathfrak{M}(q, a)$ be the Farey arc centred at a/q, $(a, q) = 1$, of the Farey dissection of order Q,

$$\mathfrak{M} = \bigcup_{q \leqslant P} \bigcup_{(a,q)=1} \mathfrak{M}(q, a) \qquad \text{and} \qquad \mathfrak{m} = \bigcup_{P < q \leqslant Q} \bigcup_{(a,q)=1} \mathfrak{M}(q, a).$$

Moreover, let

$$R(2n) = \sum_{\substack{h+k=2n \\ h,k \in [\frac{1}{4}X, 2X]}} \Lambda(h)\Lambda(k), \qquad I(2n) = \sum_{\substack{h+k=2n \\ h,k \in [\frac{1}{4}X, 2X]}} 1$$

and

$$\mathfrak{S}(2n) = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid n \\ p>2}} \left(\frac{p-1}{p-2}\right).$$

Observe that $X \ll I(2n) \ll X$ uniformly for $2n \in [X, X+H]$.

We follow the argument in **[10]**, referring to the relevant points instead of repeating the arguments. Assuming GRH, after correcting the mistake in Lemma 1 of **[10]**, see Section 5, and its consequences on the estimates which follow, from (12), (13) and (17) of **[10]** we get

$$\sum_{X \leqslant 2n \leqslant X+H} |R(2n) - I(2n)\mathfrak{S}(2n) + F(2n, X, H)|^2$$

$$\ll \sum_{X \leqslant 2n \leqslant X+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 e(-2n\alpha) d\alpha \right|^2 + X^{2+\varepsilon}, \tag{7}$$

where $F(2n, X, H)$ is a certain function satisfying

$$F(2n, X, H) \ll X \log^{-3} X \tag{8}$$

uniformly for $2n \in [X, X+H]$.

From (18) of **[10]**, see also the corrigendum, we have that

$$\sum_{X \leqslant 2n \leqslant X+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 e(-2n\alpha) d\alpha \right|^2 \ll HX \log X \max_{\substack{P<q\leqslant Q \\ (a,q)=1}} \int_{-1/qQ}^{1/qQ} \left| S\left(\frac{a}{q} + \eta\right) \right|^2 d\eta$$

and since

$$\int_{-1/qQ}^{1/qQ} \left| S\left(\frac{a}{q} + \eta\right) \right|^2 d\eta \ll \frac{\mu(q)^2}{\varphi(q)^2} \int_{-1/qQ}^{1/qQ} |T(\eta)|^2 d\eta + I(X, Q; q, a)$$

we obtain

$$\sum_{X \leqslant 2n \leqslant X+H} \left| \int_{\mathfrak{m}} S(\alpha)^2 e(-2n\alpha) d\alpha \right|^2$$

$$\ll HX \log X \max_{\substack{P<q\leqslant Q \\ (a,q)=1}} I(X, Q; q, a) + X^{2+\varepsilon}. \tag{9}$$

Hence from (5), (7) and (9) we get

$$\sum_{X\leqslant 2n\leqslant X+H} |R(2n)-I(2n)\mathfrak{S}(2n)+F(2n,X,H)|^2 \ll X^{2+\varepsilon}, \qquad (10)$$

and the estimate $E(X,H)\ll X^{\varepsilon}$ follows from (8) and (10) by a standard argument.

§3. *Proof of Theorem* 3. We first observe, as in [**9**], that writing

$$\Sigma(X,T,v)=\Sigma(X,T,v;q,a)=\sum_{\chi\ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{|\gamma|\leqslant T} X^{i\gamma}e^{iv\gamma}$$

we have

$$F(X,T;q,a)=\int_{-\infty}^{+\infty} |\Sigma(X,T,v)|^2 e^{-2|v|}dv. \qquad (11)$$

We will repeatedly use the following two lemmas. In the following, we will denote by $\alpha=\alpha(X,Q,q)$ and $\beta=\beta(X,Q,q)$ suitable real numbers, not necessarily the same at each occurrence, satisfying $c\leqslant\alpha<\beta\leqslant C$ for some absolute constants c, $C>0$.

Lemma 1. *We have*

$$\int_{\alpha X}^{\beta X} \Bigg| \sum_{\chi\ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{|\gamma|\leqslant T} y^{i\gamma}\Bigg|^2 dy \ll XF(X,T;q,a).$$

Proof. Arguing as in [**9**], we make the substitution $y=Xe^{v}$ and get

$$\int_{\alpha X}^{\beta X} \Bigg| \sum_{\chi\ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{|\gamma|\leqslant T} y^{i\gamma}\Bigg|^2 dy \ll X\int_{-\infty}^{+\infty} |\Sigma(X,T,v)|^2 e^{-2|v|}dv,$$

and Lemma 1 follows from (11).

Lemma 2. *Let* $T>U\geqslant 0$. *Then*

$$\sum_{\chi\ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{U<|\gamma|\leqslant T} X^{i\gamma} \ll T^{1/2} \max_{U\leqslant u\leqslant T} F(X,u;q,a)^{1/2}.$$

Proof. Arguing again as in [**9**] we write

$$G(v)=|\Sigma(X,T,v)-\Sigma(X,U,v)|^2.$$

From the Sobolev–Gallagher inequality, see Lemma 1.1 of Montgomery [**13**], and (11) we have

$$\left|\sum_{\chi \pmod q} \chi(a)\tau(\bar\chi) \sum_{U<|\gamma|\leqslant T} X^{i\gamma}\right|^2$$

$$= G(0) \ll \int_{-1}^{1} |G(v)|\,dv + \int_{-1}^{1} |G'(v)|\,dv$$

$$\ll F(X, T; q, a) + F(X, U; q, a) + \int_{-1}^{1} |G'(v)|\,dv. \tag{12}$$

By the Cauchy–Schwarz inequality we get

$$\int_{-1}^{1} |G'(v)|\,dv \ll \left(\int_{-1}^{1} \left|\sum_{\chi \pmod q} \chi(a)\tau(\bar\chi) \sum_{U<|\gamma|\leqslant T} X^{i\gamma} e^{iv\gamma}\right|^2 dv\right)^{1/2}$$

$$\times \left(\int_{-1}^{1} \left|\sum_{\chi \pmod q} \chi(a)\tau(\bar\chi) \sum_{U<|\gamma|\leqslant T} \gamma X^{i\gamma} e^{iv\gamma}\right|^2 dv\right)^{1/2}.$$

Hence by partial summation we obtain that

$$\int_{-1}^{1} |G'(v)|\,dv \ll T \max_{U\leqslant u\leqslant T} F(X, u; q, a) \tag{13}$$

and Lemma 2 follows from (12) and (13).

Let now $\theta \in (0, \frac{1}{2}]$ be fixed, $Q = \frac{1}{2}X^{\theta}$ and assume GMC(θ). We have

$$R(\eta; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(a)\tau(\bar\chi) W(X, \chi, \eta) + O(\log^2 qX)$$

where

$$W(X, \chi, \eta) = \sum_{\frac{1}{2}X\leqslant n\leqslant 2X} \Lambda(n)\chi(n)e(n\eta) - \delta_\chi T(\eta), \qquad \delta_\chi = \begin{cases} 1 & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}$$

Hence applying Gallagher's lemma, see Lemma 1.9 of Montgomery [**13**], using the explicit formula for L-functions, see Davenport [**2**], and writing $K = XQ^{-1/2}$, we get

$$I(X, Q; q, a) \ll \frac{X^{\varepsilon}}{q^4 Q^2}(I_1 + I_2 + I_3) + \frac{X^{1+\varepsilon}}{qQ}, \tag{14}$$

where

$$I_1=\int_{\frac{1}{4}X-h}^{\frac{1}{4}X}\left|\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{|\gamma|\leq K}\frac{(y+h)^\rho-(\frac{1}{4}X)^\rho}{\rho}\right|^2 dy,$$

$$I_2=\int_{\alpha X}^{\beta X}\left|\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{|\gamma|\leq K}\frac{(y+h)^\rho-y^\rho}{\rho}\right|^2 dy,$$

$$I_3=\int_{2X-h}^{2X}\left|\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{|\gamma|\leq K}\frac{(2X)^\rho-y^\rho}{\rho}\right|^2 dy$$

and $h=\frac{1}{2}qQ$.

We first treat I_2. We split I_2 into

$$I_2\ll I_2^-+I_2^+,$$

according to $|\gamma|\leq 2V$ and $2V<|\gamma|\leq K$, and observe that $2V<K<X$. Clearly

$$I_2^+\ll X\int_{\alpha X}^{\beta X}\left|\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{2V<|\gamma|\leq K}\frac{y^{i\gamma}}{\rho}\right|^2 dy.$$

We remove the factor $1/\rho$ by partial summation and the Cauchy–Schwarz inequality, thus getting from Lemma 1 that

$$I_2^+\ll X^{1+\varepsilon}\left(K^{-2}\int_{\alpha X}^{\beta X}\left|\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{2V<|\gamma|\leq K}y^{i\gamma}\right|^2 dy\right.$$
$$\left.+\int_V^K u^{-3}\left(\int_{\alpha X}^{\beta X}\left|\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{2V<|\gamma|\leq u}y^{i\gamma}\right|^2 dy\right)du\right)$$
$$\ll X^{2+\varepsilon}\left(K^{-2}(F(X,K;q,a)+F(X,2V;q,a))\right.$$
$$\left.+\int_{2V}^K u^{-3}(F(X,u;q,a)+F(X,2V;q,a))du\right). \qquad (15)$$

From (15) and GMC(θ) we easily see that the total contribution of I_2^+ to (14) is $\ll X^{1+\varepsilon}/qQ$.

In order to treat I_2^- we write

$$I_2^-=\int_{\alpha X}^{\beta X}\left|\int_y^{y+h}\sum_{\chi \pmod q}\chi(a)\tau(\bar{\chi})\sum_{|\gamma|\leq 2V}u^{\rho-1}du\right|^2 dy$$

and hence by the Cauchy–Schwarz inequality and Lemma 1 we get

$$I_2^- \ll \frac{h^2}{X} \int_{\alpha X}^{\beta X} \left| \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{|\gamma| \leqslant 2V} y^{i\gamma} \right|^2 dy \ll h^2 F(X, 2V; q, a). \tag{16}$$

From (16) and GMC(θ) we see that the total contribution of I_2^- to (14) is $\ll X^{1+\varepsilon}/qQ$.

Now we turn to the tails I_1 and I_3. Since their treatment is completely similar, we will treat explicitly only I_3. Again we split I_3 into

$$I_3 \ll I_3^- + I_3^+,$$

according to $|\gamma| \leqslant 2V$ and $2V < |\gamma| \leqslant K$. Moreover, the treatment of I_3^- is completely similar to the one of I_2^-, so its total contribution to (14) is, under GMC(θ), $\ll X^{1+\varepsilon}/qQ$.

In order to deal with I_3^+, we further split it into

$$I_3^+ \ll J_1 + J_2$$

with

$$J_1 \ll X \int_{\alpha X}^{\beta X} \left| \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{2V < |\gamma| \leqslant K} \frac{y^{i\gamma}}{\rho} \right|^2 dy \tag{17}$$

and

$$J_2 = \int_{2X-h}^{2X} \left| \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{2V < |\gamma| \leqslant K} \frac{(2X)^{\rho}}{\rho} \right|^2 dy$$

$$\ll hX \left| \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{2V < |\gamma| \leqslant K} \frac{(2X)^{i\gamma}}{\rho} \right|^2. \tag{18}$$

From (17) we see that J_1 can be estimated as I_2^+, and therefore its total contribution to (14) is, under GMC(θ), $\ll X^{1+\varepsilon}/qQ$.

Applying partial summation to (18), from Lemma 2, applied with $2X$ in place of X, we get that

$$J_2 \ll hX \left(\left| \frac{1}{K} \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{2V < |\gamma| \leqslant K} (2X)^{i\gamma} \right|^2 \right.$$

$$\left. + \left(\int_{2V}^{K} u^{-2} \left| \sum_{\chi \ (\mathrm{mod}\ q)} \chi(a)\tau(\bar{\chi}) \sum_{2V < |\gamma| \leqslant u} (2X)^{i\gamma} \right| du \right)^2 \right)$$

$$\ll \frac{hX}{K} \max_{2V \leqslant u \leqslant K} F(2X, u; q, a)$$

$$+ hX \left(\int_{2V}^{K} u^{-3/2} \max_{2V \leqslant v \leqslant u} F(2X, v; q, a)^{1/2} du \right)^2. \tag{19}$$

Observing that $2V \geqslant (2X)^{1-\theta}/q$, from (19) and GMC($\theta$) we see that the total contribution of J_2 to (14) is $\ll X^{1+\varepsilon}/qQ$ and Theorem 3 is proved.

§4. *Proof of Theorem* 4. Using the Sobolev–Gallagher inequality, see Lemma 1.1 of [**13**], and the Cauchy–Schwarz inequality, for $|\eta| \leqslant \xi$ we have that

$$|R(\eta; q, a)|^2 \ll \frac{1}{\xi} \int_{-\xi}^{\xi} |R(\eta; q, a)|^2 d\eta + \left(\int_{-\xi}^{\xi} |R(\eta; q, a)|^2 d\eta \right)^{1/2} \left(\int_{-\xi}^{\xi} |R'(\eta; q, a)|^2 d\eta \right)^{1/2}. \tag{20}$$

By partial summation we get

$$\int_{-\xi}^{\xi} |R'(\eta; q, a)|^2 d\eta \ll X^2 \max_{\frac{1}{4}X \leqslant Y \leqslant 2X} \int_{-\xi}^{\xi} \left| \sum_{\frac{1}{4}X \leqslant n \leqslant Y} \left(\Lambda(n) e\left(\frac{na}{q}\right) - \frac{\mu(q)}{\varphi(q)} \right) e(n\eta) \right|^2 d\eta. \tag{21}$$

The last integral in (21) can be treated exactly as $\int_{-\xi}^{\xi} |R(\eta; q, a)|^2 d\eta$, and hence we may assume without loss of generality that the maximum in (21) is attained at $Y = 2X$. We may therefore assume that

$$\int_{-\xi}^{\xi} |R'(\eta; q, a)|^2 d\eta \ll X^2 \int_{-\xi}^{\xi} |R(\eta; q, a)|^2 d\eta. \tag{22}$$

The treatment of $\int_{-\xi}^{\varepsilon} |R(\eta; q, a)|^2 d\eta$ is completely similar to the treatment of $I(X, Q; q, a)$ in the proof of Theorem 3. In fact, writing in this case $K = X(q\xi)^{1/2}$ and $h = 1/2\xi$, we get the following analogue of (14)

$$\int_{-\xi}^{\xi} |R(\eta; q, a)|^2 d\eta \ll \frac{\xi^2 X^\varepsilon}{q^2} (I_1 + I_2 + I_3) + \xi X^{1+\varepsilon}, \tag{23}$$

where the integrals I_j, $j = 1, 2, 3$, are defined after (14).

The treatment of the I_j's is formally the same as in the proof of Theorem 3, apart from the following slight difference. Write the parameter W in the definition of GMC($\theta; \xi$) as $W = W(X, q)$. Then, in this case, we split I_j into I_j^- and I_j^+ according to $|\gamma| \leqslant W(X_j, q)$ and $W(X_j, q) < |\gamma| \leqslant K$, where $X_1 = \frac{1}{4}X$, $X_2 = X$ and $X_3 = 2X$; the further splitting of I_1^+ and I_3^+ is done in the same way as in the proof of Theorem 3. Moreover, in the case when $K \geqslant X$, we may use (3) in addition to GMC($\theta; \xi$).

Following the argument leading to (15)–(19), we obtain that

$$I_j \ll \frac{X^{1+\varepsilon} q^2}{\xi}, \qquad j = 1, 2, 3,$$

uniformly for $q \leqslant X^\theta$ and $(a, q) = 1$, and hence from (23) we get

$$\int_{-\xi}^{\xi} |R(\eta; q, a)|^2 d\eta \ll \xi X^{1+\varepsilon}, \tag{24}$$

again uniformly for $q \leqslant X^\theta$ and $(a, q) = 1$.

Theorem 4 follows now from (20), (22) and (24).

§5. *Correction of* **[10]**. In this section we use the notation of **[10]**. Professor R. C. Vaughan has pointed out that the proof of Lemma 1 in **[10]** is not correct. In fact, the condition $n \in [x, x+2qQ] \cap [1, 2N]$ in (7) of **[10]**, which arises after applying Gallagher's lemma, was overlooked in what followed. This condition, which should in fact read $n \in [x, x+\frac{1}{2}qQ] \cap [1, 2N]$, gives rise to two extra integrals which are the analogues of the integrals I_1 and I_3 in Section 3 above, and the smoothing technique of Saffari–Vaughan **[15]** does not apply to such integrals. After correcting this, the bound in Lemma 1 in **[10]** has the factor L^2 replaced by L^4 in the general case of $\chi \pmod q$ with $q \leqslant Q \leqslant N$, although the bound with the factor L^2 can be shown to hold in a certain restricted range for q and Q.

As a consequence, the argument in **[10]** does not prove the results stated in the Theorem and Corollary there, which have to be modified by suitably increasing the exponent of the relevant logarithmic factors, at least when H is large. We remark that the most interesting case in **[10]** is when H is small, and in this range the argument in **[10]** suffices to prove the stated results.

However, the results in **[10]** can be retrieved by the following technical device. Instead of starting with the exponential sum $S(\alpha)$ and the counting function $R(2n)$, we start with

$$\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/N} e(n\alpha) \qquad \text{and} \qquad \tilde{R}(2n) = e^{-2n/N} R(2n)$$

and use the approximation

$$\tilde{S}\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} \frac{1}{z} + \tilde{R}(\eta; q, a)$$

where $(a, q) = 1$, $z = N^{-1} - 2\pi i \eta$ and

$$\tilde{R}(\eta; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \chi(a) \tau(\bar{\chi}) \times \left(\sum_{n=1}^{\infty} \Lambda(n) \chi(n) e^{-n/N} e(n\eta) - \frac{\delta_\chi}{z} \right) + O(\log^2 qN).$$

Following the argument in [**10**], applied to $\tilde{S}(\alpha)$ instead of $S(\alpha)$, we get the main term $2ne^{-2n/N}\mathfrak{S}(2n)$ instead of $2n\mathfrak{S}(2n)$. Moreover, the treatment of the critical error terms coming from the major arcs, as well as the treatment of the minor arcs, will depend on the analogue of the crucial Lemma 1 there, with

$$\sum_{n=1}^{\infty} \Lambda(n)\chi(n)e^{-n/N}e(n\eta) - (\delta_\chi/z)$$

instead of $\psi'(2N, \chi, \eta)$. The other error terms are easily estimated in the same way as in [**10**].

When $q=1$, the required analogue of Lemma 1 is contained in Languasco–Perelli [**11**], see (23) there. It is not difficult to see that the argument in [**11**] carries over to the general case of $\chi \pmod q$, thus providing the required analogue of Lemma 1 in [**10**], *i.e.*, under GRH we have

$$\int_{-1/qQ}^{1/qQ} \left| \sum_{n=1}^{\infty} \Lambda(n)\chi(n)e^{-n/N}e(n\eta) - \frac{\delta_\chi}{z} \right|^2 d\eta \ll \frac{NL^2}{qQ} \tag{25}$$

uniformly for $\chi \pmod q$ with $q \leqslant Q \leqslant N$. Using (25) in the same way as Lemma 1 is used in [**10**], we obtain the analogue of the Theorem in [**10**] for the modified counting function $\tilde{R}(2n)$, *i.e.*, under GRH we have

$$\sum_{N \leqslant 2n \leqslant N+H} |\tilde{R}(2n) - 2ne^{-2n/N}\mathfrak{S}(2n) + \tilde{F}(n, N, H)|^2 \ll H^{1/2}N^2L^3 \tag{26}$$

where $\tilde{F}(n, N, H)$ is a certain function satisfying

$$\tilde{F}(n, N, H) \ll NH^{-1/8}(LM)^{1/2}.$$

The Corollary in [**10**] follows at once from (26) by a standard technique.

We finally remark that, in point of principle, the mistake in Lemma 1 of [**10**] occurs also in (21) and (24) of Perelli–Pintz [**14**], and the above technical device can be successfully applied in this case too. However, since [**14**] deals only with a saving of arbitrary powers of L, simple estimates based on the orthogonality of characters suffice in this case in order to bound the extra integrals coming from the application of Gallagher's lemma, thus providing a correct proof of the results in [**14**].

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Dr. Alessandro Languasco,
Dipartimento di Matematica,
Via Dodecaneso 35,
16146 Genova, Italy.
e-mail: languasco@dima.unige.it

Professor Alberto Perelli,
Dipartimento di Matematica,
Via Dodecaneso 35,
16146 Genova, Italy.
e-mail: perelli@dima.unge.it

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