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EXPLICIT SOLUTIONS OF THE EIGENVALUE PROBLEM $-\mathrm{div}\left(\frac{Du}{|Du|}\right) = u$ IN R^2 *

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Abstract. In this paper we compute explicit solutions of the eigenvalue problem $-\mathrm{div}(Du/|Du|) = u$ in R^2, in particular explicit solutions whose truncatures are in $W^{1,1}_{\mathrm{loc}}(R^2)$, and piecewise constant ones which are sums of characteristic functions of convex sets. The solutions of the above eigenvalue problem describe the asymptotic behavior of solutions of the minimizing total variation flow. As an application, we also construct explicit solutions of the denoising problem in image processing.

Key words. eigenvalue problem, total variation flow, finite perimeter sets, denoising problem

AMS subject classifications. 35J70, 35P30, 35K65

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1. Introduction.

The main aim of this paper is to compute explicit solutions of the following eigenvalue problem:

$$-\mathrm{div}\left(\frac{Du}{|Du|}\right) = u, \quad u \in L^1_{\mathrm{loc}}(R^2). \tag{1.1}$$

Solutions to (1.1) describe the asymptotic behavior, as $t \to +\infty$, of solutions of the minimizing total variation flow in R^2 given by the equation

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(\frac{Du}{|Du|}\right) \quad \text{in } Q_T := \ ]0, T[\ \times R^2, \tag{1.2}$$

coupled with the initial condition

$$u(0) = u_0 \in L^2(R^2). \tag{1.3}$$

Indeed, as was proved in [9], if $u_0 \in L^2(R^2)$, then the solution $u(t)$ vanishes in finite time $T(u_0)$ and the rescaled function $\frac{u(t)}{T(u_0)-t}$ converges along subsequences to a solution of (1.1) as $t \to T(u_0)$; see Theorem 2.8 below. Thus, solutions of (1.1) describe the profiles of extinction of solutions of (1.2). We also notice that a solution u of (1.1) allows us to construct a solution of (1.2) of the form $v(t, x) = (1 - t)^+u(x)$.

One of the main motivations of our study comes from the total variation approach to the problems of image denoising and restoration. Indeed, as was shown in [12], solutions of (1.1) allow us to construct explicit solutions of the total variation formulation of the denoising problem [33]. Assuming that our observed image (or data) f

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comes from noisy observations of an ideal undistorted image u, the image model can be written as

$$f = u + n, \tag{1.4}$$

where n represents the noise, typically assumed to be Gaussian. In [33], Rudin, Osher, and Fatemi proposed obtaining the denoised image u by solving the constrained minimization problem

$$\text{Minimize} \int_D |Du| \quad \text{with} \quad \int_D (u-f)^2\,dx = \sigma^2 |D|, \tag{1.5}$$

where D is the image domain, typically a rectangle in R^2, and the constraint incorporates the image acquisition model given by (1.4) in terms of the variance of the noise σ^2. Let us stress here that even if three-dimensional images occur, for instance, a medical image (or video data), the case of R^2, being the case of photographs and satellite or medical images, plays an important role in image processing. In practice, problem (1.5) is solved via the unconstrained minimization problem

$$\min \left\{ \int_D |Du| + \frac{1}{2\lambda} \int_D (u-f)^2\,dx : u \in BV(D) \right\} \tag{1.6}$$

for some Lagrange multiplier $\lambda > 0$ [17]. The constraint has been introduced as a penalization term. The regularization parameter λ controls the trade-off between the goodness of fit of the constraint and the smoothness term given by the total variation. This formulation of the denoising problem pioneered the use of total variation as a regularization term and the use of bounded variation functions in image processing. The first regularization methods used the Sobolev (semi)norm $\int_D |Du|^2$ and proposed denoising the data f by solving

$$\min \left\{ \int_D |Du|^2 + \frac{1}{2\lambda} \int_D (u-f)^2\,dx : u \in W^{1,2}(D) \right\}. \tag{1.7}$$

In case $D = R^2$, the solution of (1.7) in the Fourier domain is given by

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{1 + 4\gamma\pi^2 |\xi|^2}, \quad \xi \in R^2$$

(the constants appearing in the denominator being dependent on the form of the Fourier transform). From the above formula we see that high frequencies of f (hence, the noise) are attenuated by the smoothness constraint. This was an important step, but the results were not satisfactory, mainly due to the inability of the previous functional to resolve discontinuities (edges) and oscillatory textured patterns. The smoothness constraint is too restrictive. Indeed, functions in $W^{1,2}(D)$ cannot have discontinuities along rectifiable curves. These observations motivated the introduction of total variation in image restoration models by Rudin, Osher, and Fatemi in their seminal work [33]. The a priori hypothesis is that functions of bounded variation (the BV model) [6, 24, 36] are a reasonable functional model for many problems in image processing, in particular, for restoration problems [33]. Typically, functions of bounded variation have discontinuities along rectifiable curves, being continuous in the measure theoretic sense away from discontinuities. The discontinuities could be identified with edges. The ability of this functional to describe textures is less clear;

some textures can be recovered, but up to a certain scale of oscillation. An interesting experimental discussion of the adequacy of the BV model to describe real images can be seen in [3, 29].

The analysis of problem (1.5) has been the subject of much work in the last ten years, both numerical and theoretical. It will not be our purpose to review it here, and we refer the interested reader to [10] for an account of it. Let us mention only that the existence of solutions of (1.5) for any $f \in L^2(R^N)$ follows easily from the convexity of the functional and the properties of bounded variation functions; that the equivalence between (1.5) and (1.6) was proved in [17]; and that the characterization of the Euler–Lagrange equation in distributional terms was done in [8, 12] (see [10]). To describe the behavior of solutions of (1.6), the authors started in [12] the search for explicit solutions for some particular kind of functions $f \in L^2(R^2)$. Since, when $\lambda = \Delta t$, (1.6) corresponds to the implicit in time discretization of (1.2) (also called the Crandall–Liggett scheme in semigroup theory [19]), the behavior of solutions of one of them is analogous to those of the other. This has been exploited in the papers [8, 12] (see also [10] for a full account).

In particular, in [12] we showed how the explicit solutions of (1.1) could be used to construct data $f \in L^2(R^2)$ for which we could compute the explicit solution of (1.6) in R^2. In the most simple case, if $\overline{u} \in BV(R^2)$ is a solution of (1.1) and $b \in R$, then the function $a\overline{u}$ with $a = \mathrm{sign}(b)(|b| - \lambda)^+$ is the solution of the variational problem (1.6) when $f = b\overline{u}$. In other words, the solution of (1.6) is given by the soft-thresholding rule applied to b. Other more general results were also exhibited. In particular, this established a connection with the wavelet approach to denoising given by the soft-thresholding rule applied to the wavelet coefficients of a noisy function (the uncorrupted function being in some Besov space) [20, 21, 22, 23]. In this direction, let us recall the result of Meyer [31], which proves that by applying a soft-thresholding to the coefficients of the wavelet expansion of f with respect to some orthonormal wavelet basis, one obtains a quasi-optimal solution of (1.6) in the sense that its energy is bounded by a universal constant times the actual minimum energy. Further work exploring the connection between both approaches, variational and wavelet-based, to the denoising problem can be found in [35].

Our purpose in this paper will be to make progress in the study of the solutions of the eigenvalue problem (1.1) and to derive, as a consequence, other explicit solutions of the denoising problem

$$\min\left\{\int_{R^2} |Du| + \frac{1}{2\lambda}\int_{R^2} (u-f)^2\,dx : u \in BV(R^2)\right\} \tag{1.8}$$

for some data $f \in L^2(R^2)$, $\lambda > 0$. For that, in section 2 we shall begin by recalling some preliminary facts about functions of bounded variation, a generalized Green's formula [11], and the notion of solution for the evolution equation (1.2) and for the eigenvalue problem (1.1).

In section 3 we describe the regularity properties of the level lines of the solutions of (1.1). In section 4 we study the solutions of (1.1) which are in $W^{1,1}(R^2)$, and hence do not possess discontinuities along rectifiable curves. Indeed, we compute the explicit solutions u of (1.1) whose truncatures $T_k(u) := (-k) \vee u \wedge k$ are in $W^{1,1}_{\mathrm{loc}}(R^2)$ for any $k > 0$, and we prove that the level sets $\{u > t\}$, $t > 0$ (resp., $\{u < t\}$, $t < 0$), of the nonzero solutions are balls of radius $\frac{1}{t}$ (resp., $-\frac{1}{t}$).

Then we turn our attention to the consideration of piecewise constant solutions of (1.1) which can be described as sums of characteristic functions of convex sets forming towers (or oscillating towers). As we shall prove, there are geometric restrictions on the curvature of the convex sets, as well as restrictions on their relative position to be able to combine them in towers which are solutions of (1.1). In some particular cases, this kind of geometric condition already appeared in the study of capillarity problems in domains of R^2 [18, 25, 26, 27, 28], and its analogues have also appeared in the case of crystalline variational problems [13, 14, 15]. Let us mention that consideration of convex sets is justified by the results in [12]. The analysis of piecewise constant solutions of (1.1) leads to the study of solutions of $\operatorname{div} z =$ constant in bounded and unbounded domains delimited by convex sets. Section 5 is devoted to solving the equation $\operatorname{div} z =$ constant in a bounded domain F of R^2 determined by an exterior Jordan curve ∂C_0 of class $\mathcal{C}^{1,1}$ and a finite number m of interior Jordan curves, also of class $\mathcal{C}^{1,1}$, where the unknown is a vector field $z \in L^\infty(F; R^2)$, $\|z\|_\infty \leq 1$, whose trace at the boundary is the inner or outer unit normal, depending on the Jordan curve. This is one of the basic building blocks in constructing piecewise constant explicit solutions of (1.1), the other being the solution of the equation $\operatorname{div} z = 0$ in the complement of a bounded domain made by a finite number of connected components whose boundary is a convex curve of class $\mathcal{C}^{1,1}$. This will be the purpose of section 6. By pasting together these solutions one can construct explicit piecewise constant solutions of (1.1). We shall call these solutions oscillating tower solutions of (1.1). We shall use them to construct some data $f \in L^2(R^2)$ for which the explicit solutions of (1.8) can be computed (with a soft-thresholding rule). This will be the purpose of section 8.

The solutions constructed here illustrate the behavior of solutions of (1.8), but do not exhibit all its features. The behavior of (1.8) for characteristic functions of general convex sets in R^2 (together with explicit solutions of (1.2)) was described in [1], where it was shown that the sets are eroded at high curvature points of its boundary. By the way, the extension of the above results to characteristic functions of convex sets in R^N has been started in [2]. The explicit behavior of (1.8) and (1.2) when the initial condition is the characteristic function of a general set in R^2 with smooth or piecewise smooth boundary is still to be described. We believe that with these elements on hand, one would be able to add them and produce a description of a more general class of piecewise constant solutions of (1.2). There is still a long way to go, but our explicit solutions are a first step in this direction and illustrate the behavior of soft-thresholding in some geometrically simple cases, exhibiting the role of the parameter λ in the elimination of small localized perturbations of the image (which could be assimilated to a multiple of a characteristic function of some small ball).

2. Some notation.

2.1. Functions of bounded variation and sets of finite perimeter. Let Q be an open subset of R^N. By $C_0^\infty(Q)$ (resp., $C_0^\infty(Q; R^N)$) we denote the space of functions (resp., vector fields with values in R^N) which are C^∞ and have compact support in Q.

A function $u \in L^1(Q)$ whose gradient Du in the sense of distributions is a (vector-valued) Radon measure with finite total variation in Q is called a function of bounded variation. The class of such functions will be denoted by $BV(Q)$. The total variation of Du on Q turns out to be

$$\sup\left\{\int_Q u \operatorname{div} z \, dx : z \in C_0^\infty(Q; R^N), \|z\|_{L^\infty(Q)} := \operatorname*{ess\,sup}_{x \in Q} |z(x)| \leq 1\right\} \tag{2.1}$$

(where for a vector $v = (v_1, \dots, v_N) \in R^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(Q)$ or by $\int_Q |Du|$. It turns out that the map $u \to |Du|(Q)$ is $L^1_{\rm loc}(Q)$-lower semicontinuous. $BV(Q)$ is a Banach space when endowed with the norm $\int_Q |u|\, dx + |Du|(Q)$. We recall that $BV(R^N) \subseteq L^{N/(N-1)}(R^N)$. The total variation of u on a Borel set $B \subseteq Q$ is defined as $\inf\{|Du|(A)\colon A \text{ open}, B \subseteq A \subseteq Q\}$. We denote by $BV_{\rm loc}(Q)$ the space of functions $w \in L^1_{\rm loc}(Q)$ such that $w\varphi \in BV(Q)$ for all $\varphi \in C_0^\infty(Q)$. For results and information on functions of bounded variation, we refer to [6, 24].

A measurable set $E \subseteq R^N$ is said to be of finite perimeter in Q if (2.1) is finite when u is substituted with the characteristic function χ_E of E. The perimeter of E in Q is defined as $P(E, Q) := |D\chi_E|(Q)$, and $P(E, Q) = P(R^N \setminus E, Q)$. We shall use the notation $P(E) := P(E, R^N)$. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is countably $(N-1)$ rectifiable with finite $\mathcal{H}^{N-1}$ measure, and compute the outer unit normal $\nu^E(x)$ at $\mathcal{H}^{N-1}$ almost all points x of $\partial^* E$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. Moreover, $|D\chi_E|$ coincides with the restriction of $\mathcal{H}^{N-1}$ to $\partial^* E$.

For a Lebesgue measurable subset $E \subseteq R^N$ and a point $x \in R^N$, the upper and lower densities of E at x are, respectively, defined by

$$\overline{D}(x, E) := \limsup_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|}, \quad \underline{D}(x, E) := \liminf_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|}.$$

Here $B_r(x)$ denotes the open ball of radius r centered at x and $|\cdot|$ stands for the Lebesgue measure. If the upper and lower densities are equal, their common value will be called the density of E at x, and it will be denoted by $D(x, E)$. Each set E of finite perimeter will be identified with the representative (in its Lebesgue class) given by the set of all points $x \in R^N$ such that $D(x, E) = 1$. It is clear that if ∂E is Lipschitz continuous, then the precise representative we are choosing is an open set.

If μ is a (possibly vector-valued) Radon measure and f is a Borel function, the integration of f with respect to μ will be denoted by $\int f d\mu$. When μ is the Lebesgue measure, the symbol dx will be often omitted.

By $L^1_w(]0, T[; BV(R^N))$ we denote the space of functions $v :]0, T[\to BV(R^N)$ such that $v \in L^1\left(]0, T[\times R^N\right)$, the maps $t \in]0, T[\to \int_{R^N} \phi \, dDv(t)$ are measurable for every $\phi \in C^1_0(R^N; R^N)$, and $\int_0^T |Dv(t)|(R^N)\, dt < \infty$. By $L^1_w(]0, T[; BV_{\rm loc}(R^N))$ we denote the space of functions $v :]0, T[\to BV_{\rm loc}(R^N)$ such that $v\varphi \in L^1_w(]0, T[; BV(R^N))$ for all $\varphi \in C_0^\infty(R^N)$.

If E is a subset of R^N of class $\mathcal{C}^{1,1}$, we denote by $\kappa_{\partial E}$ the ($\mathcal{H}^{N-1}$-almost everywhere defined) curvature of ∂E, nonnegative for convex sets. The following result can be proved as in [32].

Proposition 2.1. *Let $\mu \in R$ and E be a set of class $\mathcal{C}^{1,1}$. Assume that there exists an open set A such that $A \cap \partial E$ is the graph of a $\mathcal{C}^{1,1}$ function, and*

$$P(E, A) - \mu|E \cap A| \le P(E \cup B, A) - \mu|(E \cup B) \cap A| \tag{2.2}$$

for any bounded measurable set B with $\overline{B} \subset A$. Then $\kappa_{\partial E}(x) \ge \mu$ for $\mathcal{H}^{N-1}$-almost every $x \in A \cap \partial E$. Similarly, if in place of (2.2) *there holds the inequality*

$$P(E, A) - \mu|E \cap A| \le P(E \setminus B, A) - \mu|(E \setminus B) \cap A|,$$

then $\kappa_{\partial E}(x) \le \mu$ for $\mathcal{H}^{N-1}$-almost every $x \in A \cap \partial E$.

The following lemma will be used in several places. Let us include its proof for the sake of completeness.

LEMMA 2.2. *Let $A, B \subseteq R^N$ be two sets of finite perimeter such that $|A \cap B| = 0$. Then, up to a set of $\mathcal{H}^{N-1}$-measure zero, we have*

$$\partial^*(A \cup B) = (\partial^* A \setminus \partial^* B) \cup (\partial^* B \setminus \partial^* A).$$

In particular, we have

$$P(A \cup B) = P(A) + P(B) - 2\mathcal{H}^{N-1}(\partial^* A \cap \partial^* B).$$

Proof. Recall that if $E \subseteq R^N$ has finite perimeter, the essential boundary $\partial^* E$ is contained in the measure theoretic boundary $\partial^M E$ (i.e., the set of points $x \in R^N$ such that $\overline{D}(x, E) > 0$ and $\overline{D}(x, R^N \setminus E) > 0$) of E, and $\mathcal{H}^{N-1}(\partial^M E \setminus \partial^* E) = 0$ [6, 24, 36]. Let $p \in \partial^*(A \cup B)$. Then $\overline{D}(p, A \cup B) > 0$ and $\overline{D}(p, R^N \setminus (A \cup B)) > 0$. Since $R^N \setminus (A \cup B) \subseteq R^N \setminus A$ we have $\overline{D}(p, R^N \setminus A) > 0$. Similarly $\overline{D}(p, R^N \setminus B) > 0$. From $\overline{D}(p, A \cup B) > 0$, we have either $\overline{D}(p, A) > 0$ or $\overline{D}(p, B) > 0$. If $\overline{D}(p, A) > 0$ (resp., $\overline{D}(p, B) > 0$), we have $p \in \partial^* A$ (resp., $p \in \partial^* B$). Now, if $p \in \partial^* A \cap \partial^* B$, $\mathcal{H}^{N-1}$-almost everywhere, we have $D(p, A) = D(p, R^N \setminus A) = D(p, B) = D(p, R^N \setminus B) = \frac{1}{2}$. Since $|A \cap B| = 0$, we conclude

$$D(p, A \cup B) = D(p, A) + D(p, B) = \frac{1}{2} + \frac{1}{2} = 1.$$

This implies that $p \notin \partial^*(A \cup B)$, a contradiction. We conclude that $p \notin \partial^* A \cap \partial^* B$. We have proved that

$$\partial^*(A \cup B) \subseteq (\partial^* A \setminus \partial^* B) \cup (\partial^* B \setminus \partial^* A) \quad (\text{mod } \mathcal{H}^{N-1}).$$

To prove the opposite inclusion, assume that $p \in \partial^* A \setminus \partial^* B$. Then for $\mathcal{H}^{N-1}$-almost every p we may assume that

$$D(p, A) = D(p, R^N \setminus A) = \frac{1}{2}. \tag{2.3}$$

In particular, we have that $\overline{D}(p, A \cup B) > 0$. Assume that $\overline{D}(p, R^N \setminus (A \cup B)) = 0$. In this case, $D(p, A \cup B) = 1$. Using (2.3), we obtain $D(p, B) = \frac{1}{2}$. Hence, $p \in \partial^* B$, a contradiction. Thus, we also have $\overline{D}(p, R^N \setminus (A \cup B)) > 0$, and therefore $p \in \partial^*(A \cup B)$ for $\mathcal{H}^{N-1}$-almost every $p \in \partial^* A \setminus \partial^* B$. We conclude that $\partial^* A \setminus \partial^* B \subseteq \partial^*(A \cup B)$. Similarly we have that $\partial^* B \setminus \partial^* A \subseteq \partial^*(A \cup B)$. ◻

2.2. A generalized Green's formula. Let Ω be an open set in R^N. Following [11], let

$$X_2(\Omega) := \{z \in L^\infty(\Omega; R^N) : \text{div } z \in L^2(\Omega)\},$$
$$X_{2,\text{loc}}(\Omega) := \{z \in L^\infty(\Omega; R^N) : \text{div } z \in L^2_{\text{loc}}(\Omega)\}.$$

If $z \in X_{2,\text{loc}}(\Omega)$ and $w \in L^2_{\text{loc}}(\Omega) \cap BV_{\text{loc}}(\Omega)$, we define the functional (z, Dw): $C_0^\infty(\Omega) \to R$ by the formula

$$\langle (z, Dw), \varphi \rangle := -\int_\Omega w\, \varphi \,\text{div }z\, dx - \int_\Omega w\, z \cdot \nabla\varphi\, dx \qquad \forall \varphi \in C_0^\infty(\Omega). \tag{2.4}$$

Notice that

$$\langle (z, Dw), \varphi \rangle = \int_\Omega z \cdot \nabla w \, \varphi \, dx \qquad \forall w \in L^2_{\rm loc}(\Omega) \cap W^{1,1}_{\rm loc}(\Omega).$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$, then (z, Dw) is a Radon measure in Ω, and

$$\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw| \qquad \forall \text{ Borel set } B \subseteq \Omega.$$

We denote by $\theta(z, Dw) \in L^\infty_{|Dw|}(\Omega)$ the density of (z, Dw) with respect to $|Dw|$, that is,

$$(z, Dw)(B) = \int_B \theta(z, Dw) \; d|Dw| \qquad \forall \text{ Borel set } B \subseteq \Omega. \tag{2.5}$$

If $\Omega = R^N$, we have the following integration-by-parts formula [11] for $z \in X_2(R^N)$ and $w \in L^2(R^N) \cap BV(R^N)$:

$$\int_{R^N} w \text{ div } z \; dx + \int_{R^N} (z, Dw) = 0. \tag{2.6}$$

In particular, if B is bounded and has finite perimeter in R^N, from (2.6) and (2.5) it follows that

$$\int_B \text{div } z \; dx = \int_{R^N} (z, -D\chi_B) = \int_{\partial^* B} \theta(z, -D\chi_B) \; d\mathcal{H}^{N-1}. \tag{2.7}$$

Notice also that if $z_1, z_2 \in X_2(R^N)$ and $z_1 = z_2$ almost everywhere on B, then $\theta(z_1, -D\chi_B)(x) = \theta(z_2, -D\chi_B)(x)$ for $\mathcal{H}^{N-1}$-almost every $x \in \partial^* B$.

We recall the following result proved in [11].

THEOREM 2.3. *Let $\Omega \subset R^N$ be a open set with Lipschitz boundary, $1 \leq p \leq N$, $p' = \frac{p}{p-1}$. Assume that either Ω or $R^N \setminus \overline{\Omega}$ is bounded. Let $u \in BV(\Omega) \cap L^{p'}(\Omega)$ and $z \in L^\infty(\Omega; R^N)$ with* div $z \in L^p(\Omega)$. *Then, using test functions $\varphi \in C_0^\infty(\Omega)$,* (2.4) *defines a Radon measure (z, Du) in Ω, there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ such that $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega; R^N)}$, and*

$$\int_\Omega u \text{ div} z \; dx + \int_\Omega (z, Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u \; d\mathcal{H}^{N-1}.$$

In particular, if Ω or $R^N \setminus \overline{\Omega}$ is a bounded open set with Lipschitz boundary, then (2.7) has a meaning also if z is defined only on Ω and not on the whole of R^N, precisely when $z \in L^\infty(\Omega; R^N)$ with div $z \in L^1(\Omega)$. In this case we mean that $\theta(z, -D\chi_\Omega)$ coincides with $[z \cdot \nu^\Omega]$.

Remark 1. Let $\Omega \subset R^2$ be a bounded Lipschitz open set, and let $z_{\rm inn} \in L^\infty(\Omega; R^2)$ with div$z_{\rm inn} \in L^2(\Omega)$, and $z_{\rm out} \in L^\infty(R^2 \setminus \overline{\Omega}; R^2)$ with div$z_{\rm out} \in L^2(R^2 \setminus \overline{\Omega})$. Assume that

$$[z_{\rm inn} \cdot \nu^\Omega](x) = -[z_{\rm out} \cdot \nu^{R^2 \setminus \overline{\Omega}}](x) \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial\Omega.$$

Then if we define $z := z_{\rm inn}$ on Ω and $z := z_{\rm out}$ on $R^2 \setminus \overline{\Omega}$, we have $z \in L^\infty(R^2; R^2)$ and div$z \in L^2(R^2)$.

2.3. The notion of solution, and existence and uniqueness results. Consider the energy functional $\Psi : L^2(R^N) \to (-\infty, +\infty]$ defined by

$$\Psi(u) := \begin{cases} \int_{R^N} |Du| & \text{if} \quad u \in L^2(R^N) \cap BV(R^N), \\ +\infty & \text{if} \quad u \in L^2(R^N) \setminus BV(R^N). \end{cases} \tag{2.8}$$

Since the functional Ψ is convex, lower semicontinuous, and proper, then $\partial\Psi$ is a maximal monotone operator with dense domain, generating a contraction semigroup in $L^2(R^N)$ (see [16]). Therefore, we have the following result.

THEOREM 2.4. *Let $u_0 \in L^2(R^N)$. Then there exists a unique strong solution in the semigroup sense u of* (1.2), (1.3) *in* $[0,T]$ *for every $T > 0$, i.e., $u \in C([0,T]; L^2(R^N)) \cap W^{1,2}_{\rm loc}(0,T; L^2(R^N))$, $u(0) = u_0$, $u(t) \in D(\partial\Psi)$ for almost every $t \in [0,T]$, and*

$$-u'(t) \in \partial\Psi(u(t)) \quad \text{for a.e. } t \in [0,T]. \tag{2.9}$$

Moreover, if u and v are the strong solutions of (1.2) *corresponding to the initial conditions $u_0, v_0 \in L^2(\Omega)$, respectively, then*

$$\|u(t) - v(t)\|_2 \le \|u_0 - v_0\|_2 \quad \text{for any } t > 0.$$

The semigroup theory immediately provides us with existence and uniqueness results for (1.2). The characterization of $\partial\Psi$ given in Lemma 2.5 below (see [8, 9, 12] for a proof) allows us to write Theorem 2.4 in more classical terms.

LEMMA 2.5. *The following assertions are equivalent:*

(a) $(u, v) \in \partial\Psi$;

(b)

$$u \in L^2(R^N) \cap BV(R^N), \ v \in L^2(R^N), \tag{2.10}$$

$$\exists z \in X_2(R^N) \ \textit{with} \ \|z\|_\infty \le 1, \ \textit{such that} \ v = -\mathrm{div} z \quad \textit{in} \ \mathcal{D}'(R^N),$$

and

$$\int_{R^N} (z, Du) = \int_{R^N} |Du|. \tag{2.11}$$

Let us now give a more classical definition of solution for problem (1.2). As we shall notice below, this notion coincides with the notion of a strong solution in the sense of semigroups defined above.

DEFINITION 2.6. *A function $u \in C([0,T]; L^2(R^N))$ is called a strong solution of* (1.2) *if*

$$u \in W^{1,2}_{\rm loc}(0,T; L^2(R^N)) \cap L^1_w(]0,T[; BV(R^N)),$$

and there exists $z \in L^\infty\left(]0,T[\times R^N; R^N\right)$ with $\|z\|_\infty \le 1$ such that

$$u_t = \mathrm{div}\ z \qquad \textit{in} \ \mathcal{D}'(]0,T[\times R^N)$$

and

$$\int_{R^N} (z(t), Du(t)) = \int_{R^N} |Du(t)| \quad \textit{for a.e.} \ t > 0. \tag{2.12}$$

We have the following result [8, 9, 12].

THEOREM 2.7. *Let $u_0 \in L^2(R^N)$. A function $u \in C([0,T]; L^2(R^N))$ is a strong solution of* (1.2) *with $u(0) = u_0$ if and only if it is a strong solution of it in the semigroup sense. Hence there exists a unique strong solution u of* (1.2), (1.3) *in $[0,T] \times R^N$ for every $T > 0$. Moreover, if u and v are the strong solutions of* (1.2) *corresponding to the initial conditions $u_0, v_0 \in L^2(R^N)$, respectively, then*

$$\|(u(t) - v(t))^+\|_2 \leq \|(u_0 - v_0)^+\|_2 \quad \textit{for any } t > 0. \tag{2.13}$$

Obviously, using Lemma 2.5, a strong solution of (1.2) is a strong solution in the sense of semigroups. The converse implication would follow along the same lines, except for the measurability of $z(t,x)$. To ensure the joint measurability of z, one takes into account that, by the Crandall–Liggett theorem [19], semigroup solutions can be approximated by implicit-in-time discretizations of (2.9), and one constructs a function $z(t,x) \in L^\infty(]0,T[\times R^N)$ satisfying the requirements contained in Definition 2.6. For details we refer to [8, 10]. Let us finally recall that, by a suitable extension of the notion of solution, we have existence, uniqueness, and stability results with respect to convergence in $L^1_{\text{loc}}(R^N)$ for initial conditions in $L^1_{\text{loc}}(R^N)$ [12].

Theorem 2.7 can be complemented with the following result.

THEOREM 2.8. *Let $u_0 \in L^2(R^N) \cap L^N(R^N)$ with support contained in a ball B of radius $R > 0$, and let $u(t,x)$ be the unique solution of problem* (1.2). *Then $\text{supp}(u) \subseteq B$. If $T^*(u_0) = \inf\{t > 0 \colon u(t) = 0\}$, then*

$$T^*(u_0) \leq \frac{R\|u_0\|_\infty}{N}. \tag{2.14}$$

Let

$$w(t,x) := \begin{cases} \dfrac{u(t,x)}{T^*(u_0) - t} & \textit{if} \quad 0 \leq t < T^*(u_0), \\ 0 & \textit{if} \quad t \geq T^*(u_0). \end{cases}$$

Then there exists an increasing sequence $t_n \to T^*(u_0)$ and a solution $v^* \neq 0$ of the eigenvalue problem

$$v \in \partial\Psi(v) \tag{2.15}$$

such that

$$\lim_{n\to\infty} w(t_n) = v^* \quad \textit{in} \quad L^p(R^N)$$

for all $1 \leq p < \infty$.

By Lemma 2.5, equation (1.1) can be understood in more classical terms. Let us write this definition in a more general context. We shall use the truncatures $T_k(r) := (-k) \wedge r \vee k$, $r \in R$, $k > 0$.

DEFINITION 2.9. *Let Ω be an open set in R^N and let $f \in L^2_{\text{loc}}(\Omega)$. We say that a function $u \in L^1_{\text{loc}}(\Omega)$ is a solution of*

$$-\text{div}\left(\frac{Du}{|Du|}\right) = f \quad \textit{in } \Omega \tag{2.16}$$

if

$$T_k(u) \in BV_{\text{loc}}(\Omega) \qquad \forall k > 0, \tag{2.17}$$

$$\exists z \in L^\infty(\Omega; R^N) \textit{ with } \|z\|_\infty \leq 1, \textit{ such that } -\text{div} z = f \quad \textit{in } \mathcal{D}'(\Omega),$$

and

$$\langle (z, DT_k(u)), \varphi \rangle = \int_\Omega |DT_k(u)|\, \varphi \quad \textit{for any } \varphi \in C_0^\infty(\Omega), \tag{2.18}$$

where the left-hand side is defined as in (2.4).

The above definition also makes sense if we assume that $f \in L^1_{\mathrm{loc}}(R^N)$. Since this will not be needed in what follows, and to avoid cumbersome statements in subsection 2.2, we have assumed that $L^1_{\mathrm{loc}}(R^N)$.

Remark 2. If u is a solution of (2.16) and $f \in L^p_{\mathrm{loc}}(\Omega)$ with $p \geq 2$, then $(z, D\chi_{\{u>t\}}) = |D\chi_{\{u>t\}}|$ (in the sense that $\langle (z, D\chi_{\{u>t\}}), \varphi \rangle = \langle |D\chi_{\{u>t\}}|, \varphi \rangle$ for any $\varphi \in C_0^\infty(\Omega)$) for almost any $t \in R$. Indeed, by [11, Proposition 2.7], we have

$$\langle (z, DT_k(u)), \varphi \rangle = \int_{-k}^{k} \langle (z, D\chi_{\{u>t\}}), \varphi \rangle \, dt, \qquad \varphi \in C_0^\infty(\Omega),\ k > 0.$$

Since $|DT_k(u)|(\varphi) = \int_{-k}^{k} |D\chi_{\{u>t\}}|(\varphi)$, we may write (2.18) as

$$\int_{-k}^{k} \langle (z, D\chi_{\{u>t\}}), \varphi \rangle \, dt = \int_{-k}^{k} |D\chi_{\{u>t\}}|(\varphi) \, dt, \qquad \varphi \in C_0^\infty(\Omega),\ k > 0,$$

and this implies our claim.

Remark 3. If $u \in L^\infty(\Omega)$, condition (2.18) can be replaced by $(z, Du) = |Du|$.

3. Properties of L^p_{loc}-solutions. Throughout the paper, from now on we shall assume that $N = 2$.

PROPOSITION 3.1. *Let Ω be an open set in R^2, and let $u \in L^p_{\mathrm{loc}}(\Omega)$ for some $p \in]2, +\infty]$. Let u be a solution of* (1.1) *in Ω. The following assertions hold.*

(a) *If $p < +\infty$ (resp., $p = +\infty$), then for any $t \in R$ the sets $\{u > t\}$ and $\{u \geq t\}$ have boundary of class $C^{1,\alpha}$ in Ω for some $\alpha \in]0, 1[$ (resp., $\mathcal{C}^{1,1}$). Similar assertions hold for $\{u < t\}$ and $\{u \leq t\}$.*

(b) *If $u \geq a$ in Ω (resp., $u \leq a$ in Ω) for some $a \in R$, then $\kappa_{\Omega \cap \partial\{u>t\}} \geq a$ and $\kappa_{\Omega \cap \partial\{u \geq t\}} \geq a$ in the sense of distributions.*

Proof. Let us prove (a). Let t be such that $\{u > t\}$ is nonempty and has locally finite perimeter in Ω and $(z, D\chi_{\{u>t\}}) = |D\chi_{\{u>t\}}|$ (in particular, by Remark 2, for almost every t). Let E be a set of finite perimeter in R^2 such that $E \triangle \{u > t\} \subset\subset \Omega$. Take a bounded Lipschitz set Ω' with $E \triangle \{u > t\} \subset\subset \Omega' \subset \Omega$. Then, using (1.1), we have

$$\int_{\{u>t\} \cap \Omega'} \mathrm{div} z \, dx - \int_{E \cap \Omega'} \mathrm{div} z \, dx \leq P(E, \Omega') - P(\{u > t\}, \Omega'). \tag{3.1}$$

It follows that $\{u > t\}$ is a minimizer of the functional

$$P(E, \Omega) + \int_{E \cap \Omega} \mathrm{div}\ z \, dx, \qquad E \subseteq R^2, \tag{3.2}$$

with respect to perturbations with compact support in Ω. Since by assumption $-\mathrm{div} z = u \in L^p_{\mathrm{loc}}(\Omega)$ for some $p \in]2, +\infty]$, using the regularity results for prescribed curvature problems (see [7, 30]), it follows that $\Omega \cap \partial\{u > t\}$ is of class $C^{1,\alpha}$ for some $\alpha \in]0, 1[$ if $p < +\infty$, and of class $\mathcal{C}^{1,1}$ if $p = +\infty$. By the compactness property of minimizers for problem (3.2) (see, for instance, [4]) the above assertion holds for any t, and (a) follows for $\{u > t\}$.

Let us prove (b). Assume that $u \geq a$ in Ω (the case $u \leq a$ is analogous). Let $t \in R$ be such that $\{u > t\}$ is nonempty and has locally finite perimeter in Ω and $(z, D\chi_{\{u>t\}}) = |D\chi_{\{u>t\}}|$ as in Remark 2 (hence, for almost every t). Let E be a set of finite perimeter in R^2 such that $E \supseteq \{u > t\}$ and $E \setminus \{u > t\} \subset\subset \Omega' \subset \Omega$, Ω' being a bounded set with Lipschitz boundary. Then from (3.1) it follows that

$$\begin{aligned} P(\{u > t\}, \Omega') &\leq P(E, \Omega') + \int_{(E \setminus \{u>t\}) \cap \Omega'} \mathrm{div} z \, dx \\ &\leq P(E, \Omega') - a(|E \cap \Omega'| - |\{u > t\} \cap \Omega'|). \end{aligned}$$

It follows that $\{u > t\}$ is a minimizer of the functional

$$P(E, \Omega) - a|E \cap \Omega|, \qquad \{u > t\} \subseteq E \subseteq R^2,$$

with respect to perturbations with compact support in Ω. This concludes the proof of (b) [7, 30].

The corresponding assertions for the sets $\{u \geq t\}$ can be proved in a similar way. $\square$

In what follows, given a function u as in Proposition 3.1 and $t \in R$, we always identify the set $\{u > t\}$ (resp., $\{u < t\}$) with its points of density one, which is an open set. We accordingly define $\{u \geq t\}$ as the complement of $\{u < t\}$.

4. Properties of $W^{1,1}_{\mathrm{loc}}$-solutions.

PROPOSITION 4.1. *Let u be a solution of* (1.1). *Assume that $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ for some open set $\Omega \subseteq R^2$. Then for any $t \in R$ every connected component of $\Omega \cap \partial\{u > t\}$ is contained in the boundary of a ball of radius $1/t$.*

Proof. Let $t \in R$, $\gamma := \Omega \cap \partial\{u > t\}$, and $\epsilon > 0$. By Proposition 3.1 the curve γ and the two curves $\gamma_\epsilon^- := \Omega \cap \partial\{u > t - \epsilon\}$, $\gamma_\epsilon^+ := \Omega \cap \partial\{u < t + \epsilon\}$ are of class $\mathcal{C}^{1,1}$. Moreover, since $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, the two sets $\gamma_\epsilon^- \cap \gamma$ and $\gamma_\epsilon^+ \cap \gamma$ are closed sets of zero $\mathcal{H}^1$-measure. Then the curve $\gamma \setminus (\gamma_\epsilon^- \cup \gamma_\epsilon^-)$ is contained in $\Omega \cap \{|u - t| < \epsilon\}$. Since γ is of class $\mathcal{C}^{1,1}$, by (b) of Proposition 3.1 it follows that γ has curvature belonging to $(t - \epsilon, t + \epsilon)$. The thesis follows by letting $\epsilon \to 0^+$. $\square$

Note that if u is as in Proposition 4.1, then the set $\{u > t\}$ is a disjoint union of balls of radius $\frac{1}{t}$ for any $t \in R$ such that the boundary of $\{u > t\}$ is contained in Ω.

LEMMA 4.2. *Let $u \in W^{1,1}_{\mathrm{loc}}(R^2) \cap L^\infty_{\mathrm{loc}}(R^2)$ be a solution of* (1.1). *Then $u \equiv 0$.*

Proof. Assume by contradiction that $\lambda := \operatorname{ess\,sup}_{R^2} u > 0$ (the case $\operatorname{ess\,inf} u_{R^2} < 0$ can be treated in a similar way). Using Proposition 4.1 it follows that the set $\{u > t\}$ contains an open ball B_t of radius $\frac{1}{t}$ for any $t \in (0, \lambda)$. Fix $t \in (0, \lambda)$ and let $t^* := \operatorname{ess\,sup}_{B_t} u > t$. Then the closure of a connected component of the set $B_t \cap \{u = t^*\} = B_t \cap \{u \geq t^*\}$ is a closed ball $D_{t^*} \subset B_t$ of radius $\frac{1}{t^*}$. Using (1.1) we get

$$t^* = \frac{(t^*)^2}{\pi} \int_{D_{t^*}} u \, dx = -\frac{(t^*)^2}{\pi} \int_{D_{t^*}} \mathrm{div} z \, dx = 2t^*,$$

which is a contradiction. $\square$

Loosely speaking, the following proposition classifies solutions with no jumps.

PROPOSITION 4.3. *Assume that u is a solution of* (1.1) *satisfying the following assumption:*

$$\forall t \in R \;\; \exists \text{ an open set } U_t \supset \partial\{u > t\} \text{ such that } u \in L^\infty_{\mathrm{loc}}(U_t).$$

Assume also that $T_k(u) \in W^{1,1}_{\mathrm{loc}}(R^2)$ *for any* $k > 0$. *Then one of the following possibilities holds:*

- $u \equiv 0$;
- *u is positive and the set* $\{u > t\}$ *is a ball of radius* $\frac{1}{t}$ *for any* $t > 0$;
- *u is negative and the set* $\{u < t\}$ *is a ball of radius* $-\frac{1}{t}$ *for any* $t < 0$;
- *u is nonnegative,* $\{u > 0\}$ *is a halfspace, and the set* $\{u > t\}$ *is a ball of radius* $\frac{1}{t}$ *for any* $t > 0$;
- *u is nonpositive,* $\{u < 0\}$ *is a halfspace, and the set* $\{u < t\}$ *is a ball of radius* $-\frac{1}{t}$ *for any* $t < 0$;
- *both* $\{u > 0\}$ *and* $\{u < 0\}$ *are halfspaces, the set* $\{u > t\}$ *is a ball of radius* $\frac{1}{t}$ *for any* $t > 0$, *and the set* $\{u < \tau\}$ *is a ball of radius* $-\frac{1}{\tau}$ *for any* $\tau < 0$.

Proof. Assume that $\lambda := \operatorname{ess\,sup} u > 0$ (the case $\operatorname{ess\,inf} u < 0$ being similar). From Proposition 4.1 we get that $\{u > t\}$ is the disjoint union of balls of radius $\frac{1}{t}$ for any $t \in (0, \lambda)$. Reasoning as in the proof of Lemma 4.2 we deduce that $\lambda = +\infty$. Observe that, given $0 < t_1 < t_2$, to each ball $B_1 \subseteq \{u > t_1\}$ (of radius $1/t_1$) there corresponds one and only one ball $B_2 \subseteq \{u > t_2\}$ (of radius $1/t_2$) such that $B_2 \subset B_1$, and vice versa. Hence there is a pairwise correspondence between the balls of $\{u > t_1\}$ and those of $\{u > t_2\}$. Letting $t \to 0^+$, $\{u > t\}$ consists of at most two balls, since given any three disjoint balls whose radius goes to infinity, at least one of them has a distance from a fixed point which goes to infinity. Hence $u > 0$ may consist of either one halfspace, two halfspaces, or the whole of R^2.

Claim. The set $\{u > t\}$ consists of exactly one ball of radius $\frac{1}{t}$ for any $t > 0$.

Observe that, once the claim is proved, all assertions of the proposition follow, since $\{u > 0\} = \bigcup_{t>0} \{u > t\}$ can only be a halfspace or the whole of R^2. Assume by contradiction that $\{u > t\}$ is the union of two balls (of radius $\frac{1}{t}$); hence $u \geq 0$ is the union of two halfspaces of R^2. Given $\tau < 0$, the set $\{u < \tau\}$ is either empty or contains a ball of radius $-\frac{1}{\tau}$; however, by the above argument there is no place for such a ball. Hence $u \geq 0$. Then $\{u = 0\}$ is either a line or a stripe. Without loss of generality, we may assume that $\{u = 0\} = [-l, l] \times R$ for some $l \geq 0$. Let $L > l$ and, for $t > 0$ small enough and such that $(z, D\chi_{\{u>t\}}) = |D\chi_{\{u>t\}}|$, set $S_{t,L} := \{u < t\} \cap]-L, L[^2$. Since $-\mathrm{div} z = u$ is bounded in $S_{t,L}$, we have

$$
\begin{aligned}
0 \geq -\int_{S_{t,L}} u \, dx = \int_{S_{t,L}} \operatorname{div} z \, dx = \int_{\partial S_{t,L}} [z, \nu^{S_{t,L}}] \, d\mathcal{H}^1 \\
\geq \mathcal{H}^1(\partial S_{t,L} \cap \partial \{u < t\}) - \mathcal{H}^1(\partial S_{t,L} \cap \{u < t\}) \geq 4L - \mathcal{H}^1(\partial S_{t,L} \cap \{u < t\}).
\end{aligned}
$$

Letting $t \to 0^+$ and using the fact that $\{u > t\}$ is the union of two balls of radius $1/t$, we obtain $4L - 4l \leq 0$, a contradiction. Our claim is proved and the proposition follows. ∎

5. Solutions of div z = constant in bounded domains. In the following, $m \geq 1$ is an integer, and we denote by $C_0, C_1, \ldots, C_m$ bounded open sets of R^2 with boundary of class $\mathcal{C}^{1,1}$ having the following properties:

- $\overline{C_l} \subset C_0$ for any $l \in \{1, \ldots, m\}$;
- $\overline{C_l} \cap \overline{C_h} = \emptyset$ for any $l, h \in \{1, \ldots, m\}$, $l \neq h$.

We define

$$
F := C_0 \setminus \bigcup_{l=1}^{m} \overline{C_l},
$$

$$J_0 := \frac{1}{|F|}\left(\sum_{i=0}^{k} P(C_i) - \sum_{j=k+1}^{m} P(C_j)\right), \tag{5.1}$$

where $0 \le k < m$ is a fixed integer.

Given a set $E \subseteq F$ of finite perimeter in F, we also let

$$\mathcal{F}_F(E) := P(E,F) + \sum_{i=0}^{k} \mathcal{H}^1(\partial^* E \cap \partial C_i) - \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* E \cap \partial C_j) - J_0|E|.$$

Remark 4. It is clear that $\mathcal{F}_F(\emptyset) = 0$. Observe also that, thanks to the definition of J_0, $\mathcal{F}_F(F) = 0$.

We now define a class $\mathcal{A}$ of subsets of F.

DEFINITION 5.1. *Let $E \subseteq F$ be a finite perimeter set and let $J_0 > 0$. We say that $E \in \mathcal{A}$ if either $E \in \{\emptyset, F\}$ or the following conditions hold: $F \cap \partial^* E$ consists of disjoint arcs Γ of circles of radius $1/J_0$, with $\partial F \cap \overline{\Gamma} \neq \emptyset$, and*

$$\nu^E = \nu^{C_0} \quad \text{on } \overline{\Gamma} \cap \partial C_0, \tag{5.2}$$

$$\nu^E = -\nu^{C_i} \quad \text{on } \overline{\Gamma} \cap \partial C_i,\ i \in \{1,\dots,k\}, \tag{5.3}$$

$$\nu^E = \nu^{C_j} \quad \text{on } \overline{\Gamma} \cap \partial C_j,\ j \in \{k+1,\dots,m\}. \tag{5.4}$$

In (5.2), (5.3), and (5.4) we keep the notation ν^E to indicate the extension of the outer unit normal vector to ∂E at the points of $\overline{\Gamma}$.

The following result can be essentially found in [25, Theorem 1] and [26, Theorem 6.10]. Indeed, the results in [25, 26] cover the case of equalities (5.2) and (5.3), but they can be adapted to prove (5.4).

THEOREM 5.2. *Let $E \subseteq F$ be a finite perimeter set and assume that $\mathcal{F}_F(E) = \min\{\mathcal{F}_F(B) : B \subseteq F\}$. Then $E \in \mathcal{A}$.*

The equivalence (a) $\Longleftrightarrow$ (c) of the next theorem in the crystalline case has been investigated in [13].

THEOREM 5.3. *The following conditions are equivalent:*

(a) *There exists a vector field $z : F \to R^2$ satisfying*

$$z \in L^\infty(F;R^2), \quad \begin{cases} -\operatorname{div} z = J_0 & \text{in } \mathcal{D}'(F),\\ \|z\|_\infty \le 1, & \\ [z,\nu^F] = -1 & \mathcal{H}^1\text{-a.e. on } \partial C_i, \quad i \in \{0,\dots,k\},\\ [z,\nu^F] = 1 & \mathcal{H}^1\text{-a.e. on } \partial C_j, \quad j \in \{k+1,\dots,m\}. \end{cases} \tag{5.5}$$

(b) *We have*

$$J_0 \int_F w \le \int_F |Dw| + \sum_{i=0}^{k} \int_{\partial C_i} w - \sum_{j=k+1}^{m} \int_{\partial C_j} w \qquad \forall w \in BV(F). \tag{5.6}$$

(c) *For any set $E \subseteq F$ of finite perimeter in F we have $\mathcal{F}_F(E) \ge 0$.*

(d) *We have*

$$\min_{E \in \mathcal{A}} \mathcal{F}_F(E) = 0. \tag{5.7}$$

Proof. We divide the proof into several steps.

Step 1. Let Ω be an open bounded connected subset of R^2 with $\mathcal{C}^{1,1}$ boundary, $f \in L^2(\Omega)$, $g \in L^\infty(\partial\Omega)$, and $\lambda > 0$. Assume that $\|g\|_\infty < 1$. A function $u \in BV(\Omega) \subset L^2(\Omega)$ is a solution of

$$\min_{w \in BV(\Omega)} \mathcal{E}(w), \qquad \mathcal{E}(w) := \int_\Omega |Dw| + \frac{1}{2\lambda}\int_\Omega (w-f)^2\, dx - \int_{\partial\Omega} gw \tag{5.8}$$

if and only if there exists $z \in X_2(\Omega)$, with $\|z\|_\infty \le 1$, satisfying $(z, Du) = |Du|$ as measures in Ω, $[z, \nu^\Omega] = g\ \mathcal{H}^1$-almost everywhere on $\partial\Omega$ and $-\lambda \mathrm{div} z = f - u$ in $\mathcal{D}'(\Omega)$.

We observe that the functional $\mathcal{E}$ is convex and L^1-lower semicontinuous. Moreover, since $\|g\|_\infty < 1$ and $\partial\Omega$ is of class $\mathcal{C}^{1,1}$, using the results of Giusti [28] we get that $\mathcal{E}$ is coercive. Therefore it attains its minimum, which is also unique. Hence $u = \operatorname{argmin} \mathcal{E}$ if and only if $0 \in \partial\mathcal{E}(u)$, where ∂ denotes the subdifferential in L^2.

We now define the operator $\mathcal{A}_g$ in $L^2(\Omega) \times L^2(\Omega)$ as follows: $(w, v) \in \mathcal{A}_g$ if and only if $w \in BV(\Omega)$, $v \in L^2(\Omega)$, and there is a vector field $z \in L^\infty(\Omega, R^2)$ with $\|z\|_\infty \le 1$ such that $(z, Dw) = |Dw|$, $-\mathrm{div}\, z = v$ in $\mathcal{D}'(\Omega)$, and $[z, \nu^\Omega] = g\ \mathcal{H}^1$-almost everywhere on $\partial\Omega$. Let us prove that the operator $\mathcal{A}_g$ is maximal monotone. As a consequence, since $\mathcal{A}_g \subseteq \partial\mathcal{E}$ and both are maximal monotone, we conclude that $\mathcal{A}_g = \partial\mathcal{E}$. This will prove Step 1.

The monotonicity of $\mathcal{A}_g$ follows by an integration by parts. To prove the maximal monotonicity, we have to solve

$$f \in u + \mathcal{A}_g u \qquad \forall f \in L^2(\Omega). \tag{5.9}$$

First, we assume that $f \in L^\infty(\Omega)$. Let us approximate (5.9) by

$$\begin{cases} u - \mathrm{div}(\mathcal{T}_\epsilon u) = f & \text{in } \Omega, \\ [\mathcal{T}_\epsilon u, \nu^\Omega] = g & \text{in } \partial\Omega, \end{cases} \qquad \mathcal{T}_\epsilon u := \frac{Du}{\sqrt{\epsilon^2 + |Du|^2}}. \tag{5.10}$$

Following [28], we have that (5.10) has a unique solution $u_\epsilon \in BV(\Omega)$. If we further assume that $f \in W^{1,\infty}(\Omega)$, we have $u_\epsilon \in W^{1,1}(\Omega)$ (actually $u_\epsilon \in C^{2,\alpha}(\overline{\Omega})$; see [28]).

Let us prove the basic estimates required to pass to the limit as $\epsilon \to 0$.

(i) L^2 and bounded variation estimates on u_ϵ when $f \in L^\infty(\Omega)$: multiplying (5.10) by u_ϵ, after integration by parts, we get

$$\int_\Omega u_\epsilon^2 + \int_\Omega \mathcal{T}_\epsilon u_\epsilon \cdot Du_\epsilon = \int_\Omega f u_\epsilon + \int_{\partial\Omega} g u_\epsilon.$$

Since $\frac{x^2}{\sqrt{\epsilon^2 + x^2}} \ge |x| - \epsilon$ for all $x \in R$, from the above estimate we have

$$\int_\Omega u_\epsilon^2 + \int_\Omega |Du_\epsilon| \le \epsilon|\Omega| + \int_\Omega f u_\epsilon + \int_{\partial\Omega} g u_\epsilon. \tag{5.11}$$

Now, using [28, Lemma 1.2] and $\|g\|_\infty =: 1 - 2\sigma < 1$, there is a constant c depending on σ, g, Ω, such that

$$\left| \int_{\partial\Omega} gw \right| \le (1-\sigma) \int_\Omega |Dw| + c\int_\Omega |w| \qquad \forall w \in BV(\Omega). \tag{5.12}$$

Inserting (5.12) in (5.11) we obtain the estimate

$$\frac{1}{2}\int_\Omega u_\epsilon^2 + \sigma \int_\Omega |Du_\epsilon| \le (\epsilon + c^2)|\Omega| + \|f\|_2^2.$$

Thus, by extracting a subsequence, if necessary, we may assume that $u_\epsilon \to u$ in $L^p(\Omega)$ for any $1 \le p < 2$ and weakly in $L^2(\Omega)$, where $u \in BV(\Omega)$.

(ii) L^3 estimate on u_ϵ when $f \in W^{1,\infty}(\Omega)$. We multiply (5.10) by $|T_k(u_\epsilon)|u_\epsilon$. After integrating by parts we obtain

$$\int_\Omega u_\epsilon^2 |T_k(u_\epsilon)| + \int_\Omega \mathcal{T}_\epsilon u_\epsilon \cdot D(|T_k(u_\epsilon)|u_\epsilon) = \int_\Omega f|T_k(u_\epsilon)|u_\epsilon + \int_{\partial\Omega} g|T_k(u_\epsilon)|u_\epsilon.$$

Using (5.12) and

$$\int_\Omega \mathcal{T}_\epsilon u_\epsilon \cdot D(|T_k(u_\epsilon)|u_\epsilon) \geq \int_\Omega |D(|T_k(u_\epsilon)|u_\epsilon)| - \epsilon \int_\Omega [|u_\epsilon| + |T_k(u_\epsilon)|]$$

we obtain

$$\int_\Omega u_\epsilon^2 |T_k(u_\epsilon)| + \sigma \int_\Omega |D(|T_k(u_\epsilon)|u_\epsilon| \leq (\|f\|_\infty + c) \int_\Omega |T_k(u_\epsilon)||u_\epsilon| \\ + \epsilon \int_\Omega |u_\epsilon| + \epsilon \int_\Omega |T_k(u_\epsilon)|.$$

Since u_ϵ is bounded in $L^2(\Omega)$, letting $k \to \infty$, we deduce that u_ϵ is bounded in $L^3(\Omega)$. Thus also $u \in L^3(\Omega)$.

Now,

$$\int_\Omega (u_\epsilon - u)^2 \, dx \leq \left(\int_\Omega |u_\epsilon - u|^3 \, dx \right)^{1/2} \left(\int_\Omega |u_\epsilon - u| \, dx \right)^{1/2} \to 0 \quad \text{as } \epsilon \to 0.$$

Thus we may extract a sequence u_ϵ converging in $L^2(\Omega)$ to some function $u \in BV(\Omega)$. Moreover, we may assume that $\mathcal{T}_\epsilon u_\epsilon \to z$ weakly* in $L^\infty(\Omega, R^2)$. Letting $\epsilon \to 0$ in (5.10) we have

$$u - \operatorname{div} z = f \quad \text{in } \mathcal{D}'(\Omega). \tag{5.13}$$

Still we have to prove that $(z, Du) = |Du|$ and $[z, \nu^\Omega] = g$.

Let φ be a smooth function in Ω, continuous up to $\partial\Omega$. We multiply (5.10) by φ and integrate by parts to obtain

$$\int_\Omega u_\epsilon \varphi + \int_\Omega \mathcal{T}_\epsilon u_\epsilon \cdot \nabla\varphi - \int_{\partial\Omega} [\mathcal{T}_\epsilon u_\epsilon, \nu^\Omega]\varphi = \int_\Omega f\varphi. \tag{5.14}$$

Letting $\epsilon \to 0$ and using that $[\mathcal{T}_\epsilon u_\epsilon, \nu^\Omega] = g$, we obtain

$$\int_\Omega u\varphi + \int_\Omega z \cdot \nabla\varphi - \int_{\partial\Omega} g\varphi = \int_\Omega f\varphi. \tag{5.15}$$

Integrating by parts the second term of the above equality, we get

$$\int_\Omega u\varphi - \int_\Omega \operatorname{div} z \, \varphi + \int_{\partial\Omega} ([z, \nu^\Omega] - g)\varphi = \int_\Omega f\varphi. \tag{5.16}$$

Now, using (5.13) it follows that $\int_{\partial\Omega}([z, \nu^\Omega] - g)\varphi = 0$ for all test functions φ. This implies that $[z, \nu^\Omega] = g$ on $\partial\Omega$.

To prove that $(z, Du) = |Du|$, we observe that from the lower semicontinuity of $\mathcal{E}$ and the convergence $\int_\Omega (u_\epsilon - f)^2 \, dx \to \int_\Omega (u - f)^2 \, dx$ as $\epsilon \to 0$, we have

$$\begin{aligned} \int_\Omega |Du| - \int_{\partial\Omega} gu &\leq \liminf_\epsilon \left(\int_\Omega |Du_\epsilon| - \int_{\partial\Omega} gu_\epsilon \right) = \liminf_\epsilon \left(\int_\Omega (\mathcal{T}_\epsilon u_\epsilon, Du_\epsilon) - \int_{\partial\Omega} gu_\epsilon \right) \\ &= \liminf_\epsilon - \int_\Omega \operatorname{div} \mathcal{T}_\epsilon u_\epsilon \, u_\epsilon = - \int_\Omega \operatorname{div} z \, u \\ &= \int_\Omega (z, Du) - \int_{\partial\Omega} gu \leq \int_\Omega |Du| - \int_{\partial\Omega} gu. \end{aligned}$$

We conclude that $\int_\Omega (z, Du) = \int_\Omega |Du|$.

We have proved that there is a solution of (5.9) for each $f \in W^{1,\infty}(\Omega)$. Our next goal is to prove that the operator $\mathcal{A}_g$ is closed. As a consequence we obtain that (5.9) has a solution for each $f \in L^2(\Omega)$. To prove the closedness of $\mathcal{A}_g$, let $(u_n, v_n) \in \mathcal{A}_g$ be such that $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$. Then there is a vector field $z_n \in L^\infty(\Omega, R^2)$ with $\|z_n\|_\infty \le 1$ such that $v_n = -\operatorname{div} z_n$, $(z_n, Du_n) = |Du_n|$ and $[z_n, \nu^\Omega] = g$. Modulo a subsequence, we may assume that $z_n \to z$ weakly* in $L^\infty(\Omega, R^2)$ with $\|z\|_\infty \le 1$. Since $v_n = -\operatorname{div} z_n \to -\operatorname{div} z$ in $\mathcal{D}'(\Omega)$, we have $v = -\operatorname{div} z$. The proofs of the facts $[z, \nu^\Omega] = g$ and $(z, Du) = |Du|$ follow the same arguments as those in the corresponding proofs above, and we shall omit the details. We conclude that $\mathcal{A}_g$ is closed in $L^2(\Omega)$. This ends the proof that $\mathcal{A}_g$ is maximal monotone and $\partial\mathcal{E} = \mathcal{A}_g$.

Step 2. The function $u \equiv 0$ is the solution of (5.8) if and only if f and g satisfy

$$\int_\Omega |Dw| \ge \frac{1}{\lambda} \int_\Omega wf\, dx + \int_{\partial\Omega} gw \qquad \forall w \in BV(\Omega). \tag{5.17}$$

The proof follows along the same lines as the proof of [12, Lemma 1]. Clearly $u \equiv 0$ is the solution of (5.8) if and only if

$$\int_\Omega |Dw| + \frac{1}{2\lambda} \int_\Omega (w-f)^2\, dx - \int_{\partial\Omega} gw \ge \frac{1}{2\lambda} \int_\Omega f^2\, dx \qquad \forall w \in BV(\Omega). \tag{5.18}$$

Replacing w by ϵw (where $\epsilon > 0$), expanding the L^2-norm, dividing by $\epsilon > 0$, and letting $\epsilon \to 0^+$, we have (5.17).

On the other hand, if (5.17) holds, (5.18) also holds. Finally note that, replacing w by $-w$, we see that we may replace the right-hand side of (5.17) by its absolute value.

Step 3. Problem (5.5) has a solution if and only if (5.6) holds.

Note that it is enough to prove inequality (5.6) only for functions $w \in BV(F)$, which do not change sign, i.e., $w \ge 0$ or $w \le 0$.

Suppose that (5.5) has a solution z. Let $w \in BV(F)$. Multiplying $-\text{div}z = J_0$ on F by w and integrating by parts, we obtain that (5.6) holds.

Assume now that (5.6) holds. Multiplying (5.6) by $1 - \epsilon > 0$ we deduce that

$$(1-\epsilon)J_0 \int_F w \le \int_F |Dw| + (1-\epsilon) \sum_{i=0}^{k} \int_{\partial C_i} w - (1-\epsilon) \sum_{j=k+1}^{m} \int_{\partial C_j} w$$
$$\forall w \in BV(F).$$

Thus, by Step 2 with $\lambda = 1$ we deduce that $u = 0$ is a solution of (5.8) with $f = (1-\epsilon)J_0\chi_F$, and $g \equiv -(1-\epsilon)$ in ∂C_i, $i \in \{0, \dots, k\}$, and $g \equiv 1 - \epsilon$ in ∂C_j, $j \in \{k+1, \dots, m\}$, for all $\epsilon \in]0, 1[$. Then by Step 1, we know that there exists a solution $\xi_\epsilon \in L^\infty(F, R^2)$ such that $\|\xi_\epsilon\|_\infty \le 1$, $-\text{div}\xi_\epsilon = (1-\epsilon)J_0\chi_F$, $[\xi_\epsilon, \nu^F] = g$. Letting $\epsilon \to 0$, we find a vector field z satisfying (5.5).

Step 4. Conditions (b) and (c) are equivalent.

(c) follows from (b) by taking $w = \chi_E$ in (5.6) for any set of finite perimeter $E \subseteq F$. (b) follows from (c) by means of the coarea formula. Indeed, let $w \in BV(F)$, $w \ge 0$. We have

$$\begin{aligned}
J_0 \int_F w\,dx &= J_0 \int_0^\infty \int_F \chi_{\{w\geq t\}}\chi_F\,dx\,dt = J_0 \int_0^\infty |\{w \geq t\} \cap F|\,dt \\
&\leq \int_0^\infty P(\{w \geq t\}, F)\,dt + \sum_{i=0}^{k} \int_0^\infty \mathcal{H}^1(\partial^*\{w \geq t\} \cap \partial C_i)\,dt \\
&\quad - \sum_{j=k+1}^{m} \int_0^\infty \mathcal{H}^1(\partial^*\{w \geq t\} \cap \partial C_j)\,dt \\
&= \int_F |Dw| + \sum_{i=0}^{k} \int_{\partial C_i} w - \sum_{j=k+1}^{m} \int_{\partial C_j} w.
\end{aligned}$$

Let us prove the corresponding inequality for $w \in BV(F)$, $w \leq 0$. First, we observe that, writing $F \setminus E$ instead of E in (c), we obtain

$$P(F \setminus E, F) + \sum_{i=0}^{k} \mathcal{H}^1(\partial^*(F \setminus E) \cap \partial C_i) - \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^*(F \setminus E) \cap \partial C_j) - J_0|F \setminus E| \geq 0.$$

Since $P(F \setminus E, F) = P(E, F)$ and $\mathcal{H}^1(\partial^*(F \setminus E) \cap \partial C_l) = P(C_l) - \mathcal{H}^1(\partial^* E \cap \partial C_l)$, using (5.1), we may write the last equation as

$$P(E, F) + \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* E \cap \partial C_j) - \sum_{i=0}^{k} \mathcal{H}^1(\partial^* E \cap \partial C_i) + J_0|E| \geq 0. \tag{5.19}$$

Now, we may proceed as in the case where $w \geq 0$ but using (5.19) instead of (c). Indeed,

$$\begin{aligned}
J_0 \int_F w\,dx &= -J_0 \int_{-\infty}^0 \int_F \chi_{\{w\leq t\}}\chi_F\,dx\,dt = -J_0 \int_{-\infty}^0 |\{w \leq t\} \cap F|\,dt \\
&\leq \int_{-\infty}^0 P(\{w \leq t\}, F)\,dt - \sum_{i=0}^{k} \int_{-\infty}^0 \mathcal{H}^1(\partial^*\{w \leq t\} \cap \partial C_i)\,dt \\
&\quad + \sum_{j=k+1}^{m} \int_{-\infty}^0 \mathcal{H}^1(\partial^*\{w \leq t\} \cap \partial C_j)\,dt \\
&= \int_F |Dw| + \sum_{i=0}^{k} \int_{\partial C_i} w - \sum_{j=k+1}^{m} \int_{\partial C_j} w.
\end{aligned}$$

Finally, if $w \in BV(F)$, we decompose $w = w^+ + w^-$, write the corresponding inequalities (5.6) for w^+ and w^-, and add them to obtain that (5.6) holds for w.

Step 5. Condition (c) is equivalent to

$$\min_{E \subseteq F} \mathcal{F}_F(E) = \mathcal{F}_F(\emptyset) = \mathcal{F}_F(F) = 0, \tag{5.20}$$

where the minimum is taken on the sets $E \subseteq F$ of finite perimeter. Moreover, any set $E \subseteq F$ of finite perimeter minimizing the left-hand side of (5.20) belongs to $\mathcal{A}$ by Theorem 5.2; therefore condition (c) is equivalent to condition (d). ∎

Given a set $E \subseteq R^2$, of finite perimeter in R^2, we define the functional $\mathcal{G}$ as

$$\mathcal{G}(E) := P(E) - \sum_{j=k+1}^{m} P(C_j) - J_0|E \cap F|.$$

Remark 5. Recalling the definition of J_0, we have $\mathcal{G}(F \cup (\bigcup_{j=k+1}^m C_j)) = 0$.

PROPOSITION 5.4. *The following conditions are equivalent:*

(a) *The set* $F \cup (\bigcup_{j=k+1}^m C_j)$ *is a solution of the variational problem*

$$\min \left\{ \mathcal{G}(E) : \bigcup_{j=k+1}^{m} C_j \subseteq E \subseteq C_0 \setminus \bigcup_{i=1}^{k} \overline{C}_i \right\}. \tag{5.21}$$

(b) *There exists a vector field* z *satisfying* (5.5).

Remark 6. If $k = 0$ in Proposition 5.4, the last inclusion in (5.21) must be understood as $E \subseteq C_0$.

Proof of Proposition 5.4. Assume that there exists a vector field z satisfying (5.5). Given a finite perimeter set $E \subset R^2$ such that $\bigcup_{j=k+1}^m C_j \subseteq E \subseteq C_0 \setminus \bigcup_{i=1}^k \overline{C}_i$, we integrate the divergence of z on $E \cap F$ and obtain

$$\begin{aligned} J_0|E \cap F| &= -\int_{E\cap F} \operatorname{div} z \, dx \\ &\leq P(E \cap F, F) + \sum_{i=0}^{k} \mathcal{H}^1(\partial^*(E \cap F) \cap \partial C_i) - \mathcal{H}^1\left(\partial^*(E \cap F) \cap \left(\bigcup_{j=k+1}^{m} C_j\right)\right) \\ &= P(E) - \sum_{j=k+1}^{m} P(C_j). \end{aligned}$$

It follows that $\mathcal{G}(E) \geq 0$, and (a) follows.

Let us now assume that (a) holds. Let $D \subset F$ be a set of finite perimeter. By Theorem 5.3 (see condition (c)), to obtain a vector field satisfying (5.5) it is enough to prove that

$$P(D) - 2\sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* D \cap \partial C_j) \geq J_0|D|. \tag{5.22}$$

Set $A := D \cup \bigcup_{j=k+1}^m C_j$. By assumption we have

$$\begin{aligned} 0 \leq \mathcal{G}(A) &= P(A) - \sum_{j=k+1}^{m} P(C_j) - J_0|D| \\ &= P(D) - 2\sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* D \cap \partial C_j) + \sum_{j=k+1}^{m} P(C_j) - \sum_{j=k+1}^{m} P(C_j) - J_0|D|, \end{aligned}$$

which is (5.22). ◻

Remark 7. If we consider the case in which $k = 0$, then J_0 tends to zero as C_0 tends to R^2; in this case, the minimum problem (5.21) reduces to the problem considered in [12, Theorem 6].

5.1. Characterization through the curvature of the boundaries. The aim of this subsection is to prove Theorem 5.10, which is a characterization of the solvability of problem (5.5) through pointwise curvature conditions on the boundaries of the sets C_i. We begin with some preliminaries. The next definition is taken from [18, Theorem 4.1].

Definition 5.5. *Let $\Omega \subseteq R^2$ be an open set with boundary of class $\mathcal{C}^{1,1}$ and $\rho > 0$. We say that Ω satisfies the ρ-ball condition if an open ball of radius ρ can be rotated along $\partial\Omega$ in Ω in such a way that no antipods of the ball lie on $\partial\Omega$.*

It is clear that if Ω satisfies the ρ-ball condition, then it satisfies the σ-ball condition for any $\sigma \in]0, \rho]$.

Lemma 5.6. *Let $\Omega \subseteq R^2$ be an open set satisfying the ρ-ball condition for some $\rho > 0$. Then $\operatorname{ess\,sup}_{\partial\Omega} \kappa_{\partial\Omega} \leq \frac{1}{\rho}$. Moreover, given an open ball $B_\rho \subset \Omega$ of radius ρ and tangent to $\partial\Omega$, the set $\gamma \cap \partial B_\rho$ is connected for any connected component γ of $\partial\Omega$ and spans an angle strictly less than π.*

Proof. The inequality $\operatorname{ess\,sup}_{\partial\Omega} \kappa_{\partial\Omega} \leq \frac{1}{\rho}$ is immediate. Now let $p, q \in \partial B_\rho \cap \partial\Omega$, and denote by $\gamma \subset \partial B_\rho$ the shortest of the two circular arcs in ∂B_ρ having p and q as boundary points (such a γ is uniquely determined since p and q cannot be antipodal by the ρ-ball condition). If $\gamma \not\subset \partial\Omega$, we slightly rotate B_ρ along $\partial\Omega$ around p towards q, and denoting by B' such a rotated ball, one verifies that q belongs to the interior of B', thus violating the ρ-ball condition. Hence $\gamma \subseteq \partial B_\rho \cap \partial\Omega$, and γ spans an angle strictly less than π. ◻

Remark 8. In general, the inequality $\operatorname{ess\,sup}_{\partial\Omega} \kappa_{\partial\Omega} \leq \frac{1}{\rho}$ does not imply the ρ-ball condition for the set Ω. However, if Ω is a convex set with boundary of class $\mathcal{C}^{1,1}$ such that $\operatorname{ess\,sup}_{\partial\Omega} \kappa_{\partial\Omega} < \frac{1}{\rho}$, then Ω satisfies the ρ-ball condition.

Remark 9. If C_l is convex for any $l \in \{0, \ldots, m\}$, $\operatorname{ess\,sup}_{\partial C_0} \kappa_{\partial C_0} < J_0$ (in particular $J_0 > 0$), and

$$\operatorname{dist}(\partial C_l, \partial C_h) > \frac{2}{J_0} \qquad \forall (l, h) \in \{0, \ldots, m\},\ l \neq h,$$

then F satisfies the $\frac{1}{J_0}$-ball condition.

Given a function $f \in W^{1,1}(]a, b[) \cap C^{1,1}(]a, b[)$, we denote by $\kappa(x, f(x))$ the curvature of the graph of f at the point $(x, f(x))$, i.e.,

$$\kappa(x, f(x)) := -\frac{f''(x)}{(1 + f'^2(x))^{3/2}} \quad \text{for a.e. } x \in\ ]a, b[.$$

Lemma 5.7. *Let $f, g \in W^{1,1}(]a, b[) \cap C^{1,1}(]a, b[)$ be such that $f \leq g$ on $[a, b]$, and $f(a) = g(a)$, $f(b) = g(b)$. Assume that $\operatorname{ess\,inf}_{]a,b[} \kappa(x, f(x)) \geq \operatorname{ess\,sup}_{]a,b[} \kappa(x, g(x)) \geq 0$. Then $f = g$.*

Proof. By a smoothing argument we can assume that $f, g \in C^2(]a, b[)$. Suppose by contradiction that $f \neq g$, and let $c := \max_{[a,b]}(g - f) > 0$. Let us fix $\epsilon > 0$ and consider the function $f_\epsilon(x) := (1 - \epsilon) f(x/(1 - \epsilon))$, $x \in [(1 - \epsilon)a, (1 - \epsilon)b]$. Then, for ϵ small enough, the function $g - f_\epsilon$ attains its maximum at a point $\overline{x} \in]a, b[\cap](1 - \epsilon)a, (1 - \epsilon)b[$. Hence $g'(\overline{x}) = f'_\epsilon(\overline{x})$, $g''(\overline{x}) \leq f''_\epsilon(\overline{x})$. It follows that

$$\kappa(\overline{x}, g(\overline{x})) \geq \kappa(\overline{x}, f_\epsilon(\overline{x})) = \frac{1}{1 - \epsilon} \kappa\left(\frac{\overline{x}}{1 - \epsilon}, f\left(\frac{\overline{x}}{1 - \epsilon}\right)\right) > \kappa\left(\frac{\overline{x}}{1 - \epsilon}, f\left(\frac{\overline{x}}{1 - \epsilon}\right)\right),$$

which gives a contradiction. ◻

Lemma 5.8. *Let K_0 and K_1 be two bounded strictly convex sets of class $\mathcal{C}^{1,1}$ in the plane, with $K_1 \subseteq K_0$ and $K_1 \neq K_0$. Assume that $\operatorname{ess\,sup}_{\partial K_0} \kappa_{\partial K_0} \leq \operatorname{ess\,inf}_{\partial K_1} \kappa_{\partial K_1}$. Then either $\partial K_0 \cap \partial K_1 = \emptyset$ or $\partial K_0 \cap \partial K_1$ is a connected arc which spans an angle strictly less than π.*

Proof. Let Γ be a connected component of $\partial K_0 \setminus \partial K_1$, and assume that $\Gamma \neq \partial K_0$. It is enough to prove that Γ spans an angle strictly greater than π. Assume by contradiction that Γ spans an angle less than or equal to π. Then there exists an arc $\Gamma' \subset \partial K_1 \setminus \partial K_0$ which also spans an angle less than or equal to π and has the same endpoints as Γ. By the strict convexity of K_0 and K_1, and with a proper choice of a coordinate system, we may assume that Γ' and Γ are, respectively, the graphs of two functions f and g, which satisfy the assumptions of Lemma 5.7. We get a contradiction from that lemma. □

We recall the following result, which follows from [34, (6.52)].

PROPOSITION 5.9. *Let K_0 and K_1 be two bounded convex sets of class $\mathcal{C}^{1,1}$ in the plane, with $K_1 \subseteq K_0$. Assume that* $\operatorname{ess\,sup}_{\partial K_0} \kappa_{\partial K_0} \leq \operatorname{ess\,inf}_{\partial K_1} \kappa_{\partial K_1}$. *Then*

$$2\pi(|K_0| + |K_1|) - P(K_0)P(K_1) \geq 0.$$

Moreover the inequality is strict if $K_1 \subset\subset K_0$.

Remark 10. Let $\lambda > 0$. Then the function

$$\rho \to P(B_\rho) - \lambda |B_\rho| = \pi(2\rho - \lambda\rho^2)$$

attains its maximum at $\rho = 1/\lambda$.

We are now in a position to prove the main result of this section.

THEOREM 5.10. *Assume that there exists a vector field $z : F \to R^2$ satisfying* (5.5). *Then*

$$\operatorname*{ess\,sup}_{\partial C_0} \kappa_{\partial C_0} \leq J_0, \tag{5.23}$$

$$\operatorname*{ess\,inf}_{\partial C_i} \kappa_{\partial C_i} \geq -J_0, \qquad i \in \{1, \dots, k\}, \tag{5.24}$$

$$\operatorname*{ess\,inf}_{\partial C_j} \kappa_{\partial C_j} \geq J_0, \qquad j \in \{k+1, \dots, m\}. \tag{5.25}$$

Conversely, assume that

(a) *the inequality* (5.25) *holds;*

(b) $F \cup (\bigcup_{j=k+1}^m C_j)$ *satisfies the* $\frac{1}{J_0}$*-ball condition;*

(c) $\operatorname{dist}(\partial C_l, \partial C_h) > \frac{2}{J_0}$ *for all* $(l, h) \in \{0, \dots, k\}^2 \cup \{k+1, \dots, m\}^2$, $l \neq h$.

Then there exists a vector field $z : F \to R^2$ satisfying (5.5).

Remark 11. If $k = 0$ in Theorem 5.10, then condition (5.24) does not appear.

Proof of Theorem 5.10. Assume that problem (5.5) has a solution. Fix $j \in \{k+1, \dots, m\}$ and $x \in \partial C_j$. Let A be an open neighborhood of x where ∂C_j can be written as a graph; we can assume that $\overline{A} \subset C_0$ and $\overline{A} \cap (\cup_{l \in \{1,\dots,m,\}, l \neq j} C_l) = \emptyset$. We claim that

$$P(C_j) - J_0|C_j| \leq P(C_j \cup B) - J_0|C_j \cup B| \qquad \forall B \text{ Borel}, \ \overline{B} \subset A. \tag{5.26}$$

Let B be a Borel set with $\overline{B} \subseteq A$. We can assume that $P(B) < +\infty$. Define $E := B \cup \bigcup_{l=k+1}^m C_l$. Using Proposition 5.4 we have

$$0 \leq \mathcal{G}(E) = P(E) - \sum_{l=k+1}^{m} P(C_l) - J_0|E \cap F|.$$

Since $E \cap F = B \setminus C_j$ and $P(E) = P(C_j \cup B) + \sum_{l=k+1, l \neq j}^m P(C_l)$, we have

$$P(C_j \cup B) - J_0|B \setminus C_j| \geq P(C_j). \tag{5.27}$$

By substracting $J_0|C_j|$ to (5.27) we obtain (5.26). Then (5.25) is a consequence of Proposition 2.1.

Similarly, fix $x \in \partial C_0$ (resp., $x \in \partial C_i$ for some $i \in \{1, \ldots, k\}$), and let A be an open neighborhood of x where ∂C_0 (resp., ∂C_i) can be written as a graph; we can assume that $\overline{A} \cap (\cup_{l \in \{1,\ldots,m\}} C_l) = \emptyset$ (resp., $\overline{A} \subset C_0$, $\overline{A} \cap (\cup_{l \in \{1,\ldots,m\}, l \neq i} C_i) = \emptyset$). Then

$$P(C_0) - J_0|C_0| \leq P(C_0 \setminus B) - J_0|C_0 \setminus B| \tag{5.28}$$

$$(\text{resp., } P(C_i) + J_0|C_i| \leq P(C_i \cup B) + J_0|C_i \cup B|) \tag{5.29}$$

for any Borel set B with $\overline{B} \subset A$. Indeed, define $E := (F \setminus B) \cup \bigcup_{j=k+1}^m C_j$. Using Proposition 5.4 and the equality $P(E) = P(C_0 \setminus B) + \sum_{i=1}^k P(C_i)$, we have

$$\begin{aligned} 0 \leq \mathcal{G}(E) &= P(E) - \sum_{j=k+1}^m P(C_j) - J_0|E \cap F| \\ &= P(C_0 \setminus B) - P(C_0) + \sum_{i=0}^k P(C_i) - \sum_{j=k+1}^m P(C_j) - J_0|E \cap F| \\ &= P(C_0 \setminus B) - P(C_0) + J_0|F| - J_0|E \cap F|, \end{aligned}$$

where in the last equality we have used the definition of J_0. We then get

$$P(C_0) - J_0|C_0| \leq P(C_0 \setminus B) - J_0(|E \cap F| + |C_0| - |F|) = P(C_0 \setminus B) - J_0|C_0 \setminus B|,$$

which is (5.28). Then (5.23) is a consequence of Proposition 2.1.

Eventually, in the case where $x \in \partial C_i$ for some $i \in \{1, \ldots, k\}$, and A has been chosen as described above, we define again $E := (F \setminus B) \cup \left(\bigcup_{l=k+1}^m C_l\right)$. Then

$$\begin{aligned} 0 \leq \mathcal{G}(E) &= P(E) - \sum_{l=k+1}^m P(C_l) - J_0|E \cap F| \\ &= P(C_i \cup B) - P(C_i) + \sum_{l=0}^k P(C_l) - \sum_{l=k+1}^m P(C_l) - J_0|E \cap F| \\ &= P(C_i \cup B) - P(C_i) + J_0|F| - J_0|E \cap F|, \end{aligned}$$

which implies

$$\begin{aligned} P(C_i) + J_0|C_i| &\leq P(C_i \cup B) + J_0(|F| - |E \cap F| + |C_i|) \\ &= P(C_i \cup B) + J_0|C_i \cup B|, \end{aligned} \tag{5.30}$$

and, by Proposition 2.1, (5.24) follows.

Assume now that (a)–(c) hold. Notice that condition (b) implies (5.23), which, in turn, implies $J_0 > 0$. Observe also that, by (5.25), the sets $C_{k+1}, \ldots, C_m$ are strictly convex.

Denote by $E_{\min} \in \mathcal{A}$ a solution of the minimum problem (5.7), with $E_{\min} \notin \{\emptyset, F\}$. By Theorem 5.3 and Remark 4, it is enough to prove that

$$\mathcal{F}_F(E_{\min}) \geq 0. \tag{5.31}$$

We can assume that $E_{\min}$ is connected, since the functional $\mathcal{F}_F$ is additive on connected components [5]. Recall that, by the definition of $\mathcal{A}$, the closure of (any connected component of) $E_{\min}$ must intersect ∂F.

Let Γ be a connected component of $F \cap \partial E_{\min}$, and let p, q be the endpoints of Γ, with $p \in \partial C_j$ and $q \in \partial C_i$, for some $i, j \in \{0, \dots, m\}$ (p not necessarily different from q). We recall that Γ meets tangentially ∂F (see conditions (5.2)–(5.4)) and, by assumption (b), Γ is contained in the boundary of an open ball $B \subseteq F \cup (\bigcup_{n=k+1}^m C_n)$ of radius $\frac{1}{J_0}$. We now divide the proof into three steps. We first show that p and q cannot belong both to the same ∂C_i when $i \leq k$.

Step 1. If $i \leq k$, then $i \neq j$.

Assume by contradiction that $i = j \leq k$. Using assumption (b), by Lemma 5.6 (applied with $\Omega := F \cup (\bigcup_{l=k+1}^m C_l)$) it follows that p and q are the extrema of an arc $\gamma \subseteq \partial B \cap \partial C_i$ which spans an angle strictly less than π. Notice that $\partial B = \gamma \cup \Gamma$; moreover, recalling that the curvature (which is equal to $1/J_0$) of $E_{\min}$ inside F is positive, either $E_{\min} = B$ or $E_{\min} = B \setminus C_{\overline{j}}$ for some index $\overline{j} \geq k+1$. Observe that, in the latter case, $C_{\overline{j}} \subset\subset B$ and, by condition (c), there cannot be any other C_l, with $l \geq k+1$ and $l \neq \overline{j}$, with $C_l \subseteq B$. Let us consider a new set $E' := B' \setminus B$ if $E_{\min} = B$ (resp., $E' := B' \setminus C_{\overline{j}}$ if $E_{\min} = B \setminus C_{\overline{j}}$), where B' is a ball obtained by slightly translating B towards the interior of F, and slightly modifying its radius. By Remark 10 we have $\mathcal{F}_F(E') < \mathcal{F}_F(E_{\min})$, which contradicts the minimality of $E_{\min}$. We now show that either $i \leq k$ and $j \geq k+1$ or vice versa.

Step 2. The cases $i, j \leq k$ and $i, j \geq k+1$ cannot happen.

By assumption (c) and Step 1 it is clear that the case $i, j \leq k$ cannot happen, nor can the case $i, j \geq k+1$ with $i \neq j$. We have to exclude the case $i = j \geq k+1$. Recalling that (5.25) implies the strict convexity of C_j, using (a) and (5.4), we have that $C_j \subseteq B$. Using again the strict convexity of C_j, Lemma 5.8 implies that $\partial C_j \cap \partial B$ is a connected arc which spans an angle strictly less than π. Hence we get a contradiction by slightly modifying $E_{\min}$ as in Step 1.

By Steps 1 and 2 we conclude that there exists an arc Γ of $F \cap \partial E_{\min}$ whose endpoints p, q satisfy $p \in \partial C_j$, $q \in \partial C_i$, and $i \in \{0, \dots, k\}$, $j \in \{k+1, \dots, m\}$.

In the following, we write C_i for $i \leq k$, but we mean $R^2 \setminus \overline{C_0}$ when $i = 0$.

Let us call the inner (resp., outer) side of Γ the side of Γ inside (resp., outside) $E_{\min}$. Notice that from conditions (5.2)–(5.4) C_i cannot lie in the inner side of Γ and C_j cannot lie in the outer side of Γ. Moreover, since $J_0 > 0$ the inner (resp., outer) side of Γ is also the side of Γ inside (resp., outside) B.

Step 3. We have $B = E_{\min} \cup C_j$.

Let $p' \in \partial C_j$ be the endpoint of an arc $\Gamma' \subseteq F \cap \partial E_{\min}$. Then Γ' is contained in the boundary of an open ball $B' \subseteq F \cup \left(\bigcup_{l=k+1}^m C_l\right)$ of radius $\frac{1}{J_0}$. By assumption (c) Γ' cannot meet another set $C_{j'}$ with $j' \geq k+1$, $j' \neq j$. On the other hand, the above discussion implies $C_j \subseteq B'$. Let us suppose that the other endpoint q' of Γ' (different from p') belongs to $\partial C_{i'}$ for some $i' \leq k$. Observe that $B' \cap C_{i'} = \emptyset$. If $i \neq i'$, then $B \neq B'$ (if $B = B' \ni \{q, q'\}$, then $\mathrm{dist}(C_i, C_{i'}) \leq \frac{2}{J_0}$, a contradiction with assumption (c)). Since B and B' contain C_j, we have $B \cap B' \neq \emptyset$. Now, Γ is an arc of ∂B joining $p \in \overline{B} \cap \overline{B'}$ to $q \in \partial C_i \cap \overline{B}$, $q \notin \overline{B'}$, whereas Γ' is joining $p' \in \overline{B} \cap \overline{B'}$ to $q' \in \partial C_{i'} \cap \overline{B'}$, $q \notin \overline{B}$. It follows that either $\Gamma \cap \Gamma' \neq \emptyset$ or there exists another arc of $\partial B \cap \partial E_{\min} \cap F$ different from Γ intersecting Γ'; see Figure 5.1. Since these arcs intersect transversally, this contradicts the fact that $\partial E_{\min}$ is smooth. It follows that $i = i'$. Moreover, since $(B \cup B') \cap C_i = \emptyset$, for the same reason (i.e., the fact that $\partial E_{\min}$ is smooth) we also get $B = B'$.

The ball B cannot meet, nor contain, any other set $C_{i'}$ with $i' \leq k$, $i \neq i'$, nor any other set $C_{j'}$ with $j' \geq k+1$, $j' \neq j$. Thus $B = E_{\min} \cup C_j$ (see Figure 5.2) and Step 3 is proved.

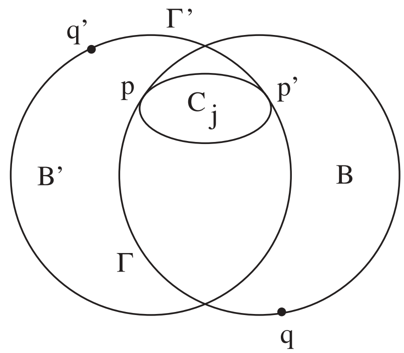

FIG. 5.1. *The two intersecting balls B and B'.*

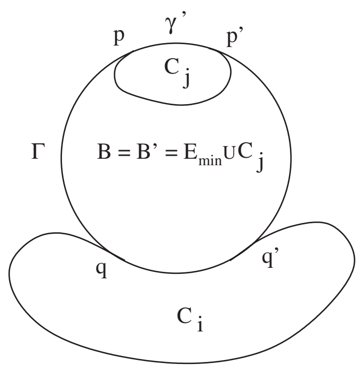

FIG. 5.2. *The minimizing set $E_{\min}$ ($i \geq 1$).*

We now conclude the proof. Applying Proposition 5.9 with $K_1 = C_j$ and $K_0 = B$, we compute (see Figure 5.2)

$$\begin{aligned}\mathcal{F}_F(E_{\min}) &= P(E_{\min}, F) + \mathcal{H}^1(\partial C_i \cap \partial E_{\min}) - \mathcal{H}^1(\partial C_j \cap \partial E_{\min}) - J_0|E_{\min}| \\ &= \frac{2\pi}{J_0} - P(C_j) - J_0\left(\frac{\pi}{J_0^2} - |C_j|\right) = \frac{\pi}{J_0} - P(C_j) + J_0|C_j| \geq 0,\end{aligned}$$

which gives (5.31) and hence the thesis. ∎

PROPOSITION 5.11. *Let K_0, K_1 be two bounded open convex sets of R^2 with boundary of class $\mathcal{C}^{1,1}$ such that $\overline{K_1} \subseteq K_0$. Let $F := K_0 \setminus \overline{K_1}$. Let*

$$J := \frac{P(K_0) - P(K_1)}{|F|} > 0.$$

If

$$\operatorname*{ess\,sup}_{\partial K_0} \kappa_{\partial K_0} \leq J, \tag{5.32}$$

$$\operatorname*{ess\,inf}_{\partial K_1} \kappa_{\partial K_1} \geq J, \tag{5.33}$$

then there exists a vector field $z \in L^\infty(F, R^2)$ *with* $\|z\|_\infty \leq 1$ *such that*

$$\begin{cases} -\operatorname{div} z = J & \text{in } \mathcal{D}'(F), \\ [z, \nu^F] = -1 & \mathcal{H}^1\text{-a.e. on } \partial K_0, \\ [z, \nu^F] = 1 & \mathcal{H}^1\text{-a.e. on } \partial K_1. \end{cases} \tag{5.34}$$

Remark 12. Proposition 5.11 admits a direct proof along the lines of [27]. Notice also that, thanks to Remark 9, Proposition 5.11 would be a consequence of Theorem 5.10 (in the case $k = 0$ and $m = 1$) if the strict inequality in (5.32) were valid.

Proof of Proposition 5.11. Let us prove that assumptions (a) and (b) of Theorem 5.10 hold for $F_\lambda := K_{0\lambda} \setminus \overline{K_{1\lambda}}$, where $K_{0\lambda} := (1+\lambda)K_0$, $K_{1\lambda} := (1-\lambda)K_1$, $\lambda > 0$ being small enough. We observe that $P(K_{0\lambda}) = (1+\lambda)P(K_0)$, $P(K_{1\lambda}) = (1-\lambda)P(K_1)$, $|K_{0\lambda}| = (1+\lambda)^2|K_0|$, and $|K_{1\lambda}| = (1-\lambda)^2|K_1|$; hence

$$J_\lambda := \frac{P(K_{0\lambda}) - P(K_{1\lambda})}{|F_\lambda|} = J + \frac{\lambda}{|F|}(P(K_0) + P(K_1) - 2J(|K_0| + |K_1|)) + o(\lambda).$$

Since

$$\operatorname*{ess\,sup}_{\partial K_{0\lambda}} \kappa_{\partial K_{0\lambda}} = \frac{1}{1+\lambda} \operatorname*{ess\,sup}_{\partial K_0} \kappa_{\partial K_0} \leq \frac{1}{1+\lambda} J,$$

it suffices to prove that $\frac{1}{1+\lambda} J < J_\lambda$ to conclude that

$$\operatorname*{ess\,sup}_{\partial K_{0\lambda}} \kappa_{\partial K_{0\lambda}} < J_\lambda. \tag{5.35}$$

By Remark 8, this implies that $K_{0\lambda}$ satisfies the $\frac{1}{J_\lambda}$-ball condition. Now, $\frac{1}{1+\lambda} J < J_\lambda$ for λ small enough if and only if

$$2P(K_0)|K_1| < P(K_1)(|K_0| + |K_1|). \tag{5.36}$$

Since $\overline{K_1} \subset K_0$, using Proposition 5.9 and the isoperimetric inequality, we deduce

$$|K_0| + |K_1| > \frac{1}{2\pi} P(K_0)P(K_1) \geq 2\frac{P(K_0)|K_1|}{P(K_1)},$$

and we obtain (5.36), and therefore also (5.35).

Let us prove that condition (b) of Theorem 5.10 holds. Since

$$\operatorname*{ess\,inf}_{\partial K_{1\lambda}} \kappa_{\partial K_{1\lambda}} = \frac{1}{1-\lambda} \operatorname*{ess\,inf}_{\partial K_1} \kappa_{\partial K_1} \leq \frac{1}{1-\lambda} J,$$

to conclude that

$$\operatorname*{ess\,sup}_{\partial K_{1\lambda}} \kappa_{\partial K_{1\lambda}} \geq J_\lambda, \tag{5.37}$$

it suffices to prove that $\frac{1}{1-\lambda} J \geq J_\lambda$. Now, $\frac{1}{1-\lambda} J > J_\lambda$ for λ small enough if and only if

$$2P(K_1)|K_0| < P(K_0)(|K_0| + |K_1|). \tag{5.38}$$

Again, since $\overline{K_1} \subseteq K_0$, using Proposition 5.9 and the isoperimetric inequality, we deduce

$$|K_0| + |K_1| > \frac{1}{2\pi} P(K_0)P(K_1) \geq 2\frac{P(K_1)|K_0|}{P(K_0)},$$

and we conclude that (5.38), and therefore also (5.37), holds.

By Theorem 5.10, there exists a vector field $z_\lambda \in L^\infty(F_\lambda, R^2)$ such that $\|z_\lambda\|_\infty \leq 1$, satisfying

$$\begin{cases} -\operatorname{div} z_\lambda = J_\lambda & \text{in } \mathcal{D}'(F_\lambda), \\ [z_\lambda, \nu^{F_\lambda}] = -1 & \mathcal{H}^1\text{-a.e. on } \partial K_{0\lambda}, \\ [z_\lambda, \nu^{F_\lambda}] = 1 & \mathcal{H}^1\text{-a.e. on } \partial K_{1\lambda}. \end{cases}$$

Letting $\lambda \to 0^+$ we obtain a solution of (5.34). $\square$

6. Solutions of div $z = 0$ in an unbounded domain. In this section we assume that $C_0 = R^2$, $k \geq 1$, we let $C_1, \dots, C_m$ be as in section 5, and we let $F := R^2 \setminus \bigcup_{i=1}^m \overline{C}_i$. We are concerned with the existence of a vector field $z : F \to R^2$ such that

$$\text{(6.1)}\quad z \in L^\infty(F, R^2), \quad \begin{cases} -\operatorname{div} z = 0 & \text{in } \mathcal{D}'(F), \\ \|z\|_\infty \leq 1, & \\ [z, \nu^F] = -1 & \mathcal{H}^1\text{-a.e. on } \partial C_i,\ i \in \{1, \dots, k\}, \\ [z, \nu^F] = 1 & \mathcal{H}^1\text{-a.e. on } \partial C_j,\ j \in \{k+1, \dots, m\}. \end{cases}$$

THEOREM 6.1. *The following conditions are equivalent:*

(i) *Problem* (6.1) *has a solution.*

(ii) *We have*

$$\text{(6.2)} \qquad 0 \leq \int_F |Dw| + \sum_{i=1}^k \int_{\partial C_i} w - \sum_{j=k+1}^m \int_{\partial C_j} w \qquad \forall w \in BV(F).$$

(iii) *For any $E \subseteq F$ of finite perimeter, we have*

$$\text{(6.3)} \qquad P(E, F) \geq \left| \sum_{j=k+1}^m \mathcal{H}^1(\partial^* E \cap \partial C_j) - \sum_{i=1}^k \mathcal{H}^1(\partial^* E \cap \partial C_i) \right|.$$

(iv) *Let E_1 be a solution of the variational problem*

$$\text{(6.4)} \qquad \min \left\{ P(E) : \bigcup_{j=k+1}^m C_j \subseteq E \subseteq R^2 \setminus \bigcup_{i=1}^k C_i \right\}.$$

Then we have

$$\text{(6.5)} \qquad P(E_1) = \sum_{j=k+1}^m P(C_j).$$

Let E_2 be a solution of the variational problem

$$\text{(6.6)} \qquad \min \left\{ P(E) : \bigcup_{i=1}^k C_i \subseteq E \subseteq R^2 \setminus \bigcup_{j=k+1}^m C_j \right\}.$$

Then we have

$$\text{(6.7)} \qquad P(E_2) = \sum_{i=1}^k P(C_i).$$

Remark 13. Notice that (iv) implies that each C_l is a convex set. Moreover, since any minimizer of problems (6.4) and (6.6) has boundary (lying inside F) made of a finite number of segments which intersect tangentially ∂F (and there are only a finite number of such segments), the number of such minimizers is finite. Finally, conditions (6.5) and (6.7) are essentially distance conditions between sets C_i of the same type; for example, they are satisfied if $\mathrm{dist}(\partial C_i, \partial C_j) > P(C_l)$ for any $(i,j,l) \in \{1,\dots,k\}^3 \cup \{k+1,\dots,m\}^3$, $i \neq j$.

Proof. We divide the proof into four steps.

Step 1. Let $f \in L^2(F)$, $g \in L^\infty(\partial F)$, $\lambda > 0$. The following hold:

(a) Assume that $\|g\|_\infty < 1$. The function u is the solution of

$$\text{(6.8)} \quad \min_{w\in BV(F)} Q(w), \qquad Q(w) := \int_F |Dw| + \frac{1}{2\lambda}\int_F (w-f)^2\, dx - \int_{\partial F} gw\, d\mathcal{H}^1$$

if and only if there exists $z \in X_2(F)$ with $\|z\|_\infty \leq 1$ satisfying $(z, Du) = |Du|$ as measures in F, $[z, \nu^F] = g$ $\mathcal{H}^1$-almost everywhere on ∂F and $-\lambda \mathrm{div} z = f - u$ in $\mathcal{D}'(F)$.

(b) The function $u \equiv 0$ is the solution of (6.8) if and only if

$$\int_F |Dw| \geq \frac{1}{\lambda}\int_F wf\, dx - \int_{\partial F} gw \qquad \forall w \in BV(F).$$

Let us prove both assertions. Let $R > 0$ be such that $R^2 \setminus F \subset\subset B_R = B_R(0)$. We consider the functional

$$\text{(6.9)} \qquad Q_R(w) := \int_{B_R\cap F} |Dw| + \frac{1}{2\lambda}\int_{B_R\cap F} (w-f)^2\, dx - \int_{\partial F} gw\, d\mathcal{H}^1$$

defined for $w \in BV(B_R \cap F)$. Now, since $\|g\|_\infty < 1$ and ∂F is of class $\mathcal{C}^{1,1}$, using the results of Giusti [28], we know that the convex functional Q_R is lower semicontinuous and proper, and it attains its infimum in $BV(B_R \cap F)$. Let $w_n \to w$ in $L^2(B_R \cap F)$. Then $Q_R(w) \leq \liminf_n Q_R(w_n) \leq \liminf_n Q(w_n)$. Since this is true for all $R > 0$, we deduce that $Q(w) \leq \liminf_n Q(w_n)$. Thus, Q is convex, lower semicontinuous, and proper. As we shall note below, Q attains its infimum in $BV(F)$. Hence $u = \mathrm{argmin}\, Q$ if and only if $0 \in \partial Q(u)$.

Now, we define the operator $\mathcal{A}'_g$ in $L^2(F) \times L^2(F)$ as follows: $(w, v) \in \mathcal{A}'_g$ if and only if $w \in BV(F)$, $v \in L^2(F)$, and there is a vector field $z \in L^\infty(F, R^2)$ with $\|z\|_\infty \leq 1$ such that $(z, Dw) = |Dw|$ and $-\mathrm{div}\, z = v$ in $\mathcal{D}'(F)$, $[z, \nu^F] = g$ $\mathcal{H}^1$-almost everywhere on ∂F. We claim that the operator $\mathcal{A}'_g$ is maximal monotone. The monotonicity of $\mathcal{A}'_g$ follows by an integration by parts. To prove the maximal monotonicity we have to solve the equation

$$\text{(6.10)} \qquad f \in u + \mathcal{A}'_g u \qquad \forall f \in L^2(F).$$

First, we assume that $f \in L^p(F)$ for any $p \in [1, \infty]$. Let us approximate (6.10) by

$$\text{(6.11)} \qquad \begin{cases} u - \mathrm{div}\, z = f & \text{in } \mathcal{D}'(B_R \cap F), \\ [z, \nu^{B_R\cap F}] = g & \mathcal{H}^1\text{-a.e. in } \partial F, \\ [z, \nu^{B_R\cap F}] = 0 & \mathcal{H}^1\text{-a.e. in } \partial B_R, \end{cases}$$

where $z \in L^\infty(B_R \cap F, R^2)$ is such that $\|z\|_\infty \leq 1$ and $(z, Du) = |Du|$. Then, by Step 1 of the proof of Theorem 5.3, equation (6.11) has a unique solution u_R. Let z_R

denote the associated vector field. Let us comment on the basic estimates required to pass to the limit as $R \to \infty$.

(i) L^2 and bounded variation estimates on u_R: multiplying (6.11) by u_R, after integration by parts, we get

$$\int_{B_R \cap F} u_R^2 + \int_{B_R \cap F} |Du_R| = \int_{B_R \cap F} f u_R + \int_{\partial F} g u_R. \tag{6.12}$$

Now, using [28, Lemma 2.2], there exists $\epsilon_0 > 0$ such that, for each $\delta > 0$, there is $c(\delta) > 0$ such that

$$\left| \int_{\partial F} gw \right| \leq \left(1 - \frac{\epsilon_0}{2}\right) \int_{S_\delta} |Dw| + c(\delta) \int_{S_\delta} |w| \qquad \forall w \in BV(B_R \cap F), \tag{6.13}$$

where $S_\delta := \{x \in B_R \cap F : \mathrm{dist}(x, \partial F) < \delta\}$, where the constant $c(\delta)$ does not depend on $R > 0$. Using (6.13) in (6.12) we obtain the estimate

$$\frac{1}{4} \int_{B_R \cap F} u_R^2 + \epsilon_0 \int_{B_R \cap F} |Du_R| \leq \frac{1}{2} \|f\|_2^2 + C|S_\delta|.$$

Thus, by extracting a subsequence, if necessary, we may assume that $u_R \to u$ in L^p_{loc} for any $1 \leq p < 2$ and weakly in $L^2(F)$ where $u \in L^2(F)$ and $\int_F |Du| < \infty$.

Let us mention that, as a consequence of (6.13), if $Q(u_n)$ is bounded, we obtain that $\int_F |u_n|^2$ and $\int_F |Du_n|$ are bounded and, therefore, Q attains its infimum.

(ii) L^p estimate on u_R: let $\eta_p : R \to R$ be a smooth function such that $\eta_p'(r) > 0$ for all $r \in R$, $\eta_p(0) = 0$, and $\mathrm{sign}(r)\eta_p(r)$ behaves as $|r|^{p-1}$ as $r \to \infty$. We multiply (6.11) by $\eta_p(u_R)$. Integrating by parts and using (6.13), we obtain

$$\int_{B_R \cap F} u_R \eta_p(u_R) \leq \int_{B_R \cap F} |f||\eta_p(u_R)| + c(\delta) \int_{S_\delta} |\eta_p(u_R)|. \tag{6.14}$$

Let $p = 1$, and assume that $|\eta_1(r)| \leq 1$ for any $r \in R$. We obtain

$$\int_{B_R \cap F} u_R \eta_1(u_R) \leq \int_{B_R \cap F} |f| + c(\delta)|S_\delta|. \tag{6.15}$$

Take a sequence $\eta_{1,n}(r)$ such that $\eta_{1,n}(r) \to \mathrm{sign}(r)$ for any $r \neq 0$. Using Fatou's theorem we deduce that

$$\int_{B_R \cap F} |u_R| \leq \int_{B_R \cap F} |f| + c(\delta)|S_\delta|.$$

Assume that u_R is bounded in L^q. Using $p = q$ in (6.14) and proceeding in the same way, we deduce that u_R is bounded in L^{q+1}. This implies that u_R is bounded in L^p for all $p < \infty$. Thus $u \in L^p(F)$ for any $1 \leq p < \infty$.

Now, let $R > M > 0$, where M is such that all sets C_i are contained in $B_{M/4}(0)$. Let $\varphi \in W^{1,\infty}(R^2)$ be such that $\varphi = 0$ on $B_{M/2}(0)$, $\varphi = 1$ outside $B_M(0)$, and it increases linearly along the rays from 0 to 1 in $B_M(0) \setminus B_{M/2}(0)$. We multiply (6.11) by $u_R \varphi^2$ and integrate by parts to obtain

$$\int_{B_R \cap F} u_R^2 \varphi^2 + \int_{B_R \cap F} |Du_R| \varphi^2 = \int_{B_R \cap F} f u_R \varphi^2 - \int_{B_R \cap F} u_R z_R \cdot \nabla(\varphi^2).$$

Hence

$$\int_{B_R\cap F} u_R^2\varphi^2 \le \int_{B_R\cap F} |f||u_R|\varphi^2 + \int_{B_R\cap F} |u_R|\varphi|\nabla\varphi|$$
$$\le \frac{1}{2}\int_{B_R\cap F} |f|^2\varphi^2 + \frac{1}{2}\int_{B_R\cap F} |u_R|^2\varphi^2 + \|u_R\varphi\|_{3/2}\|\nabla\varphi\|_3.$$

As $|\nabla\varphi| \le \frac{2}{M}$ we have

$$\|\nabla\varphi\|_3 \le \frac{2}{M}\left(\frac{3}{4}\pi M^2\right)^{1/3} \le \frac{C}{M^{1/3}}.$$

Since $\|u_R\varphi\|_{3/2}$ is bounded independently of R and M, we have

$$\int_{B_R\cap F} u_R^2\varphi^2 \le C\int_{B_R\cap F} |f|^2\varphi^2 + \frac{C}{M^{1/3}}.$$

Thus, given $\epsilon > 0$ we find M large enough so that

$$\int_{B_R\cap F} u_R^2\varphi^2 \le \epsilon$$

for any $R > M$. Assume that u_R is extended by 0 outside B_R. Thus u_R is equi-integrable near infinity. Thus, to prove that $u_R \to u$ in $L^2(F)$ it is sufficient to prove that $u_R \to u$ in $L^2_{\rm loc}(F)$. For that, let $\varphi \in C_0^\infty(R^2)$. Then

$$\int_F |u_R - u|^2\varphi^2 \le \left(\int_F |u_R - u|^3\varphi^2\right)^{1/2}\left(\int_F |u_R - u|\varphi^2\right)^{1/2} \to 0 \quad \text{as } R\to\infty,$$

since the first integral is bounded independently of R and the second tends to 0 as $R \to \infty$.

The two previous estimates imply that we may extract a subsequence u_R converging in $L^2(F)$ to some function $u \in BV(F)$. Moreover, we may assume that $z_R \to z$ weakly* in $L^\infty(F, R^2)$. Letting $R \to \infty$ in (6.11) we have

$$u - \operatorname{div} z = f \quad \text{in } \mathcal{D}'(F). \tag{6.16}$$

We still have to prove that $(z, Du) = |Du|$ and $[z, \nu^F] = g$.

Let φ be a smooth function in F, continuous up to ∂F and vanishing for large values of $|x|$. We multiply (6.11) by φ and integrate by parts to obtain

$$\int_{B_R\cap F} u_R\varphi + \int_{B_R\cap F} z_R\cdot\nabla\varphi - \int_{\partial F}[z_R, \nu^{B_R\cap F}]\varphi = \int_{B_R\cap F} f\varphi. \tag{6.17}$$

Letting $R \to \infty$ and using that $[z_R, \nu^{B_R\cap F}] = g$, we obtain

$$\int_F u\varphi + \int_F z\cdot\nabla\varphi - \int_F g\varphi = \int_F f\varphi. \tag{6.18}$$

Integrating by parts the second term of the above equality, we get

$$\int_F u\varphi - \int_F \operatorname{div} z\,\varphi + \int_{\partial F}([z, \nu^F] - g)\varphi = \int_F f\varphi. \tag{6.19}$$

Using (6.16) it follows that $\int_{\partial F}([z,\nu^F]-g)\varphi = 0$ for any φ. This implies that $[z,\nu^F]=g$ on ∂F. To prove that $(z,Du)=|Du|$, we observe that from the lower semicontinuity of Q and the convergence $\int_{B_R\cap F}(u_R-f)^2\,dx \to \int_F(u-f)^2\,dx$ as $R\to\infty$, we have

$$\begin{aligned}
&\int_F |Du| - \int_{\partial F} gu \le \liminf_R \left(\int_{B_R\cap F}|Du_R| - \int_{\partial F} gu_R\right)\\
&\quad = \liminf_R \left(\int_{B_R\cap F}(z_R,Du_R) - \int_{\partial F} gu_R\right) = \liminf_R -\int_{B_R\cap F}\operatorname{div} z_R\, u_R = -\int_F \operatorname{div} z\, u\\
&\quad = \int_F (z,Du) - \int_{\partial F} gu \le \int_F |Du| - \int_{\partial F} gu.
\end{aligned}$$

We conclude that $\int_F(z,Du)=\int_F|Du|$.

We have proved that there is a solution of (6.10) for each $f\in L^\infty(F)\cap L^2(F)$. Our next purpose is to prove that the operator $\mathcal{A}'_g$ is closed. As a consequence we obtain that (6.10) has a solution for any $f\in L^2(B_R\cap F)$. To prove the closedness of $\mathcal{A}'_g$, let $(u_n,v_n)\in\mathcal{A}'_g$ be such that $(u_n,v_n)\to(u,v)$ in $L^2(F)\times L^2(F)$. Then there is a vector field $z_n\in L^\infty(F,R^2)$ with $\|z_n\|_\infty\le 1$ such that $v_n=-\operatorname{div} z_n$, $(z_n,Du_n)=|Du_n|$ and $[z_n,\nu^F]=g$. Up to a subsequence, we may assume that $z_n\to z$ weakly* in $L^\infty(F,R^2)$ with $\|z\|_\infty\le 1$. Since $v_n=-\operatorname{div} z_n\to -\operatorname{div} z$ in $\mathcal{D}'(F)$, we have $v=-\operatorname{div} z$. The proofs of the facts $[z,\nu^F]=g$ and $(z,Du)=|Du|$ follow the same arguments as the corresponding proofs in Theorem 5.3, and we shall omit the details. We conclude that $\mathcal{A}'_g$ is closed.

Since $\mathcal{A}'_g\subseteq\partial Q$ and both are maximal monotone, we conclude that $\mathcal{A}'_g=\partial Q$. This proves (a).

The proof of (b) follows along the same lines as the proof of Step 2 in Theorem 5.3.

Step 2. (i) $\Longleftrightarrow$ (ii). Note that, as before, we may replace the condition "$\forall w\in BV(F)$" by "$\forall w\in BV(F)$ such that $w\ge 0$ or $w\le 0$."

Suppose that (6.1) has a solution z. Let $w\in BV(F)$. Multiplying (6.1) by w and integrating by parts, we obtain (6.2).

Assume now that (6.2) holds for any $w\in BV(F)$. Multiplying (6.2) by $(1-\epsilon)$, we deduce that

$$0\le \int_F |Dw| + (1-\epsilon)\sum_{i=1}^{k}\int_{\partial C_i} w - (1-\epsilon)\sum_{j=k+1}^{m}\int_{\partial C_j} w \qquad \forall w\in BV(F).$$

Using Step 1(b), we deduce that $u\equiv 0$ is a solution of (6.8) with $f=0$, and $g\equiv 1-\epsilon$ on ∂C_j and $g\equiv -(1-\epsilon)$ on ∂C_i for all $\epsilon>0$. Then by Step 2(a), we know that there exists a solution $\xi_\epsilon\in L^\infty(F,R^2)$ such that $\|\xi_\epsilon\|_\infty\le 1$, $-\operatorname{div}\xi_\epsilon=0$, $[\xi_\epsilon,\nu^F]=g$. Letting $\epsilon\to 0$, we find a vector field z satisfying (6.1).

The equivalence between (ii) and (iii) can be proved in the same manner as the equivalence between (b) and (c) in Theorem 5.3 was, and we shall omit the details.

Step 3. (iii)$\Rightarrow$(iv). Let $X := E_1\setminus\bigcup_{j=k+1}^m C_j$. Using (iii) we have

$$\sum_{j=k+1}^{m}\mathcal{H}^1(\partial^* X\cap\partial C_j) - \sum_{i=1}^{k}\mathcal{H}^1(\partial^* X\cap\partial C_i)\le P(X,F). \tag{6.20}$$

Using Lemma 2.2, we have

$$P(E_1)=P\left(X\cup\bigcup_{j=k+1}^{m} C_j\right) = P(X)+\sum_{j=k+1}^{m}P(C_j) - 2\mathcal{H}^1\left(\partial^* X\cap\left(\bigcup_{j=k+1}^{m}\partial C_j\right)\right).$$

Then, using (6.20), we have

$$\begin{aligned}
P(E_1) &= P(X) + \sum_{j=k+1}^{m} P(C_j) - 2\sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j) \\
&= P(X,F) + \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j) + \sum_{i=1}^{k} \mathcal{H}^1(\partial^* X \cap \partial C_i) \\
&\quad + \sum_{j=k+1}^{m} P(C_j) - 2\sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j) \\
&= P(X,F) + \sum_{i=1}^{k} \mathcal{H}^1(\partial^* X \cap \partial C_i) + \sum_{j=k+1}^{m} P(C_j) - \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j) \\
&\geq \sum_{j=k+1}^{m} P(C_j).
\end{aligned}$$

The proof for the set E_2 is analogous.

Step 4. (iv)⇒(iii). Let $X \subseteq F$ be a set of finite perimeter. Let E_1 be a minimizer of (6.4) and set $D := \bigcup_{j=k+1}^{m} C_j$. Using (6.5) and the minimality of E_1, we have

$$\sum_{j=k+1}^{m} P(C_j) = P(E_1) \leq P\left(X \cup D\right). \tag{6.21}$$

Using Lemma 2.2 and (6.21), we have

$$\begin{aligned}
P\left(X \cup D\right) &= P(X) + P(D) - 2\mathcal{H}^1(\partial D \cap \partial^* X) \\
&\leq P(X) + P(X \cup D) - 2\mathcal{H}^1(\partial D \cap \partial^* X).
\end{aligned}$$

Hence

$$\begin{aligned}
2\sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j) \leq P(X) = P(X,F) &+ \sum_{i=1}^{k} \mathcal{H}^1(\partial^* X \cap \partial C_i) \\
&+ \sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j).
\end{aligned}$$

We then have

$$\sum_{j=k+1}^{m} \mathcal{H}^1(\partial^* X \cap \partial C_j) \leq P(X,F) + \sum_{i=1}^{k} \mathcal{H}^1(\partial^* X \cap \partial C_i).$$

The other inequality follows by considering the set E_2 and using condition (6.6) instead of (6.4). ∎

7. Examples of solutions of the eigenvalue problem (1.1). Let us give an example of how, by pasting the solutions of problems (5.5) and (6.1), we can construct solutions of the eigenvalue problem (1.1).

Let C_i, $i = 1, \ldots, m$, $1 \le k \le m$, be a family of convex sets of class $\mathcal{C}^{1,1}$ satisfying the conditions in section 5. For each $i \in \{1, \ldots, m\}$ let us consider $C_{i1}, C_{i2}, \ldots, C_{im_i}$ open bounded sets with boundary of class $\mathcal{C}^{1,1}$ with the following properties:

- $\overline{C_{ij}} \subset C_i$ for any $j \in \{1, \ldots, m_i\}$;
- $\overline{C_{ij}} \cap \overline{C_{ij'}} = \emptyset$ for any $j, j' \in \{1, \ldots, m_i\}$, $j \neq j'$.

For $i \in \{1, \ldots, m\}$ we define

$$F_i := C_i \setminus \bigcup_{j=1}^{m_i} \overline{C_{ij}}, \qquad J_i := \frac{\sum_{j=0}^{k_i} P(C_{ij}) - \sum_{j=k_i+1}^{m_i} P(C_{ij})}{|F_i|},$$

where $k_i \in \{1, \ldots, m_i\}$ are given. Assume that

(a) $\operatorname{ess\,inf}_{\partial C_{ij}} \kappa_{\partial C_{ij}} \ge J_i$, $i \in \{1, \ldots, m\}$, $\ j \in \{k_i + 1, \ldots, m_i\}$;

(b) $F_i \cup (\bigcup_{j=k_i+1}^{m_i} \overline{C_{ij}})$ satisfies the $\frac{1}{J_i}$-ball condition for any $i \in \{1, \ldots, m\}$;

(c) $\operatorname{dist}(\partial C_{ij}, \partial C_{ij'}) > \frac{2}{J_i}$, $i \in \{1, \ldots, m\}$, $(j, j') \in \{0, \ldots, k_i\}^2 \cup \{k_i+1, \ldots, m_i\}^2$, $j \neq j'$, where we have denoted $C_{i0} = C_i$;

(d)

$$\operatorname*{ess\,sup}_{\partial C_{ij}} \kappa_{\partial C_{ij}} \le \frac{P(C_{ij})}{|C_{ij}|} =: J_{ij}, \qquad i \in \{1, \ldots, m\}, \ j \in \{1, \ldots, m_i\}.$$

Notice that $J_i > 0$, since (b) implies $\operatorname{ess\,sup}_{\partial C_{i0}} \kappa_{\partial C_{i0}} \le J_i$, and also

$$\operatorname*{ess\,inf}_{\partial C_{ij}} \kappa_{\partial C_{ij}} \ge -J_i, \qquad j \in \{1, \ldots, k_i\}.$$

Using Theorems 5.10 and 6.1, together with [12, Theorem 4], we have the existence of vector fields $\xi_{\text{ext}} \in L^\infty(R^2 \setminus \bigcup_{i=1}^m C_i)$, $\xi_i \in L^\infty(F_i)$, $\xi_{ij} \in L^\infty(C_{ij})$, such that $\|\xi_{\text{ext}}\|_\infty \le 1$, $\|\xi_i\|_\infty \le 1$, $\|\xi_{ij}\|_\infty \le 1$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$, satisfying

$$\text{(7.1)} \qquad \begin{cases} -\operatorname{div} \xi_{\text{ext}} = 0 & \text{on } R^2 \setminus \bigcup_{i=1}^m C_i, \\ [\xi_{\text{ext}}, \nu^{R^2 \setminus \cup_{i=1}^m C_i}] = -1 & \mathcal{H}^1\text{-a.e. on } \partial C_i, \ i \in \{1, \ldots, k\}, \\ [\xi_{\text{ext}}, \nu^{R^2 \setminus \cup_{i=1}^m C_i}] = 1 & \mathcal{H}^1\text{-a.e. on } \partial C_j, \ j \in \{k+1, \ldots, m\}, \end{cases}$$

$$\text{(7.2)} \quad \begin{cases} -\operatorname{div} \xi_i = J_i & \text{on } F_i, \\ [\xi_i, \nu^{F_i}] = -1 & \mathcal{H}^1\text{-a.e. on } \partial C_{ij}, \ j \in \{0, \ldots, k_i\}, \\ [\xi_i, \nu^{F_i}] = 1 & \mathcal{H}^1\text{-a.e. on } \partial C_{ij}, \ j \in \{k_i + 1, \ldots, m_i\}, \end{cases} \qquad i \in \{1, \ldots, m\},$$

$$\text{(7.3)} \quad \begin{cases} -\operatorname{div} \xi_{ij} = \dfrac{P(C_{ij})}{|C_{ij}|} & \text{on } C_{ij}, \\ [\xi_{ij}, \nu^{C_{ij}}] = -1 & \mathcal{H}^1\text{-a.e. on } \partial C_{ij}, \end{cases} \qquad i \in \{1, \ldots, m\}, \ j \in \{1, \ldots, m_i\}.$$

Now, we may paste together these vector fields to define $\xi \in L^\infty(R^2, R^2)$, $\|\xi\|_\infty \le 1$, by

$$\xi := \begin{cases} \xi_{\text{ext}} & \text{on } R^2 \setminus \bigcup_{i=1}^m C_i, \\ -\xi_i & \text{on } F_i, \ i = 1, \ldots, k, \\ \xi_i & \text{on } F_i, \ i = k+1, \ldots, m, \\ \xi_{ij} & \text{on } C_{ij}, \ i = 1, \ldots, k, \ j = 1, \ldots, k_i, \\ -\xi_{ij} & \text{on } C_{ij}, \ i = 1, \ldots, k, \ j = k_i + 1, \ldots, m_i, \\ -\xi_{ij} & \text{on } C_{ij}, \ i = k+1, \ldots, m, \ j = 1, \ldots, k_i, \\ \xi_{ij} & \text{on } C_{ij}, \ i = k+1, \ldots, m, \ j = k_i + 1, \ldots, m_i, \end{cases}$$

satisfying

$$-\mathrm{div}\,\xi = \begin{cases} 0 & \text{on } R^2 \setminus \bigcup_{i=1}^m C_i, \\ -J_i & \text{on } F_i,\ i=1,\dots,k, \\ J_i & \text{on } F_i,\ i=k+1,\dots,m, \\ J_{ij} & \text{on } C_{ij},\ i=1,\dots,k,\ j=1,\dots,k_i, \\ -J_{ij} & \text{on } C_{ij},\ i=1,\dots,k,\ j=k_i+1,\dots,m_i, \\ -J_{ij} & \text{on } C_{ij},\ i=k+1,\dots,m,\ j=1,\dots,k_i, \\ J_{ij} & \text{on } C_{ij},\ i=k+1,\dots,m,\ j=k_i+1,\dots,m_i. \end{cases}$$

Thus, if we define

$$\begin{aligned} u := &-\sum_{i=1}^{k} J_i\chi_{F_i} + \sum_{i=k+1}^{m} J_i\chi_{F_i} + \sum_{i=1}^{k}\sum_{j=1}^{k_i} J_{ij}\chi_{C_{ij}} - \sum_{i=1}^{k}\sum_{j=k_i+1}^{m_i} J_{ij}\chi_{C_{ij}} \\ &- \sum_{i=k+1}^{m}\sum_{j=1}^{k_i} J_{ij}\chi_{C_{ij}} + \sum_{i=k+1}^{m}\sum_{j=k_i+1}^{m_i} J_{ij}\chi_{C_{ij}}, \end{aligned}$$

then u is a solution of (1.1). Therefore, by pasting solutions of problems like (7.1), (7.2), (7.3), we may construct solutions of (1.1).

8. Some explicit solutions of the denoising problem. The previous results allow us to explicitly compute the minimum of the denoising problem (1.8) for some data $f \in L^2(R^2)$. Let us recall that a vector field $z \in X_2(R^2)$ with $\|z\|_\infty \le 1$ satisfying

$$-\mathrm{div}\, z = F \in L^2(R^2)$$

exists if and only if [31, 12]

$$\|F\|_* := \sup\left\{ \left| \int_{R^2} Fv\, dx \right| : v \in BV(R^2),\ \int_{R^2} |Dv| \le 1 \right\} \le 1.$$

PROPOSITION 8.1. *Let $u_i \in BV(R^2)$, $u_i \ge 0$, be such that $u_i \wedge u_j = 0$, $i,j \in \{1,\dots,m\}$, $i \ne j$. Assume that u_i and $\sum_{i=1}^m u_i$ are solutions of the eigenvalue problem (1.1), $i \in \{1,\dots,m\}$. Let $b_i \in R$, $i=1,\dots,m$, and $f := \sum_{i=1}^m b_i u_i$. Also let $\lambda > 0$. Then the solution u of the variational problem (1.8) is $u := \sum_{i=1}^m \mathrm{sign}(b_i)(|b_i| - \lambda)^+ u_i$.*

Observe that if (*) $\sum_{i=1}^m u_i$ is a solution of (1.1), then (**) $\|\sum_{i=1}^m u_i\|_* \le 1$. Notice that, using (8.2) below, it is easy to prove that both conditions (*) and (**) are, indeed, equivalent.

Proof. Under our assumptions we have $u_i \in BV(R^2) \subset L^2(R^2)$, $i=1,\dots,m$, and hence $f \in L^2(R^2)$. Recall that a function $u \in BV(R^2)$ is the solution of (1.8) if and only if u is the solution of

$$u - \lambda \mathrm{div}\left(\frac{Du}{|Du|}\right) = f. \tag{8.1}$$

Observe that since each u_i is a solution of (1.1), multiplying (1.1) by u_i and integrating by parts, we obtain

$$\int_{R^2} u_i^2\, dx = \int_{R^2} |Du_i|. \tag{8.2}$$

Let us prove that $u = \sum_{i=1}^m \mathrm{sign}(b_i)(|b_i| - \lambda)^+ u_i$ is the solution of (8.1). Let $I_\lambda := \{i \in \{1, \dots, m\} : |b_i| \geq \lambda\}$, $H_\lambda := \{i \in \{1, \dots, m\} : |b_i| < \lambda\}$. Since, in this case,

$$f - u = \lambda \sum_{i \in I_\lambda} \mathrm{sign}(b_i) u_i + \sum_{i \in H_\lambda} b_i u_i,$$

to prove that u is a solution of (8.1) we have to construct a vector field $\xi \in L^\infty(R^2; R^2)$ with $\|\xi\|_\infty \leq 1$, such that

$$-\mathrm{div}\,\xi = \sum_{i \in I_\lambda} \mathrm{sign}(b_i) u_i + \sum_{i \in H_\lambda} \frac{b_i}{\lambda} u_i \tag{8.3}$$

and $(\xi, Du) = |Du|$. Let $F \in L^2(R^2)$ denote the right-hand side of (8.3), and let $F^+ := \sup(F, 0)$, $F^- := \sup(-F, 0)$. Let us prove that $\|F\|_* \leq 1$. In order to prove this, we let $v \in BV(R^2)$. Since

$$\int_{R^2} F v\, dx \leq \int_{R^2} (F^+ v^+ + F^- v^-)\, dx$$

and $\int_{R^2} |Dv| = \int_{R^2} |Dv^+| + \int_{R^2} |Dv^-|$, the inequality $\int_{R^2} F v\, dx \leq \int_{R^2} |Dv|$ follows if we prove that

$$\int_{R^2} F^+ v^+\, dx \leq \int_{R^2} |Dv^+| \quad \text{and} \quad \int_{R^2} F^- v^-\, dx \leq \int_{R^2} |Dv^-|.$$

Thus, without loss of generality, we may assume that $F \geq 0$ (i.e., all b_i appearing in the definition of F are nonnegative) and $v \geq 0$. Then, using that $\frac{b_i}{\lambda} \leq 1$ for any $i \in H_\lambda$, we have that

$$0 \leq F \leq \sum_{i=1}^m u_i.$$

Since, by assumption, $\| \sum_{i=1}^m u_i \|_* \leq 1$, we have

$$\int_{R^2} F v\, dx \leq \int_{R^2} \sum_{i=1}^m u_i v\, dx \leq \int_{R^2} |Dv|.$$

Therefore $\|F\|_* \leq 1$. Thus, there is a vector field $\xi \in L^\infty(R^2; R^2)$ such that $\|\xi\|_\infty \leq 1$, satisfying (8.3).

As $(|b_i| - \lambda)^+ = 0$ for all $i \in H_\lambda$, we have

$$\int_{R^2} |Du| = \sum_{i \in I_\lambda} (|b_i| - \lambda) \int_{R^2} |Du_i|.$$

Since $u_i \wedge u_j = 0$ for any $i, j \in \{1, \dots, m\}$, $i \neq j$, then $Fu = \sum_{i \in I_\lambda} (|b_i| - \lambda) u_i^2$, and we have

$$\int_{R^2} (\xi, Du) = -\int_{R^2} \mathrm{div}\,\xi\, u\, dx = \int_{R^2} F u\, dx = \sum_{i \in I_\lambda} (|b_i| - \lambda) \int_{R^2} u_i^2\, dx;$$

applying (8.2) we obtain

$$\int_{R^2} (\xi, Du) = \sum_{i \in I_\lambda} (|b_i| - \lambda) \int_{R^2} |Du_i|\, dx = \int_{R^2} |Du|,$$

which in turn implies that $(\xi, Du) = |Du|$, since $\|\xi\|_\infty \leq 1$. ∎

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